Useful Identities

- $\Phi(z) \rightleftarrows \text{erf}$

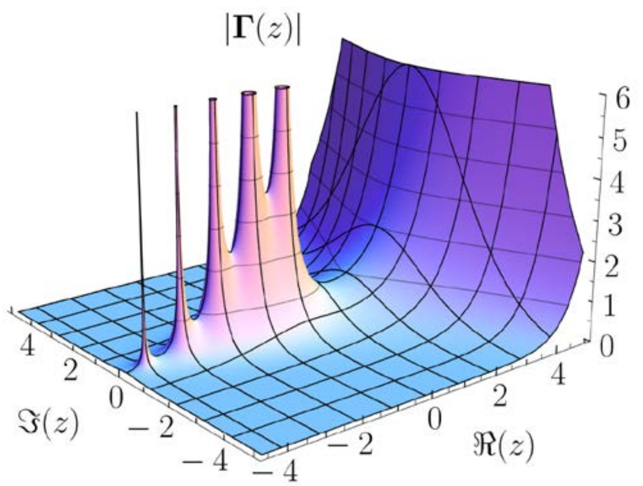
For the error function and for c.d.f. of Gaussian distribution, using $\Phi(-z) = 1 - \Phi(z)$:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = 2\Phi(x\sqrt{2}) - 1 \qquad \Phi(z) = \frac{1}{2}\left(1 + erf\left(\frac{z}{\sqrt{2}}\right)\right)$$

- $\int e^{-x^2} dx$ and modulus

$$\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi} \qquad \int_{-\infty}^{\infty} e^{-x^2/2} = \sqrt{2\pi} \qquad \int_{-\infty}^{\infty} e^{-2x^2} = \sqrt{\frac{\pi}{2}}$$

- The Factorial (Gamma) Function and its identities. Note that in full generality the function is in fact a solution to an extended complex integral (a "meromorphic[1]" function); our familiar improper integral is only defined for arugment with real part >*0*.



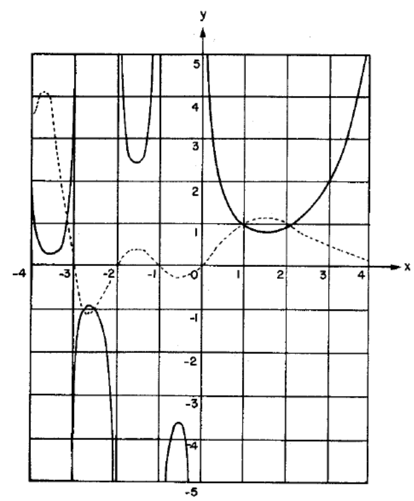


FIGURE 6.1. *Gamma function.* *

———, $y=\Gamma(x)$, – – – –, $y=1/\Gamma(x)$

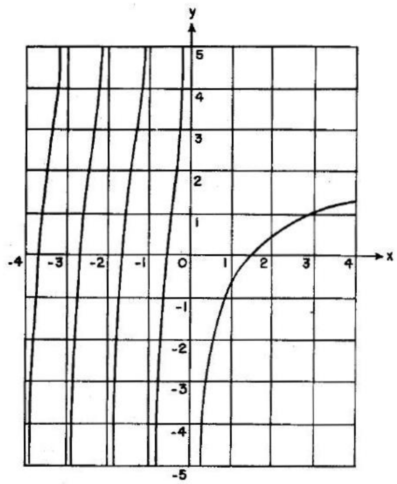
$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt\,,\ \text{Re}(\alpha) > 0\,.$$

$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) = \alpha! = \alpha(\alpha-1)!$, giving $\Gamma(\alpha) = (\alpha-1)!$ and $\Gamma(\alpha+\nu) = (\alpha+\nu-1)!$.

The Digamma function is $\frac{d}{dx}\ln\Gamma(z) \triangleq \psi(z) = \frac{\dot{\Gamma}(z)}{\Gamma(z)}$. Recurrence relations include

$\psi(z+1) = \psi(z) + \frac{1}{z}$ and $\psi(1-z) = \psi(z) + \pi\cot\pi z$.

[1] See http://en.wikipedia.org/wiki/Meromorphic_function.



Plot of digamma function

$\dot{\psi}(z)$ is the trigamma function. The Beta function is related to $\Gamma(z)$:

$$B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-t}\,dx=\int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\,dt\text{, or } B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

These are normalizing constants used for pdf kernels.

Landau Order Notation

Landau order notation is a convenient way to summarize remainder terms of sequences or expansions (or tail events when dealing with random phenomena) for use in truncations; is does not give an exact statement of the remainder error, but rather an approximation to this error which may then be used to simplify other calculations and results.

We will consider the case in R^n; suppose we have sequences of real numbers a_n and b_n; we will also be considering sequences of random vectors X_n, Y_n, etc.

Deterministic Order Notation

Stochastic Order Notation

Mathematical Spaces

Space – A set X

Algebra or Topology

Topological Space

Measureable Space

Probability Space

Linear Space

Metric Space

Vector Space

Normed Linear Space

Inner Product Space

Examples: vector spaces of functions

- $C_b(x)$: Bounded, continuous functions on $\mathcal{X}$

Sets

Lim sup A_n= [A_n i.o.]

An i.o. == Mn>=1Um>=nAm. No matter how large n, there m>=n and omega in Am.

Lim inf

Outcome eventiallly there is an n s.t. for any m>=n, omega is in Am. For for this n and up, omega is in An.

Ex, due to Florescu: toss coin infinitely many time, 1 come up infimintely many times.

Convergence and Continuity

Convergence of sequences[2]

Convergence of sequences

Uniform continuity

$\varepsilon, \delta_\varepsilon$ are not functions of x $\forall x$.

Convergence of sequence of functions - Pointwise

$f_n, f, g \in$ functional space $\mathcal{X}$, typically a metric, normed linear, or inner product space. W.s. $f_n \xrightarrow{pw} f$ as $n \to \infty$, or $\lim f_n(t) = f(t)$; i.e., $\forall t \in \mathcal{X}$ and $\forall \varepsilon, n_\varepsilon > 0$, $\left\| f_n(t) - f(t) \right\| < \varepsilon \;\; \forall n > n_\varepsilon$. Procedure: fix t and check the convergence of the sequence of values of $f_n(t)$ to $f(t)$.

[2] Handy.Analysis.wmf

Convergence of sequence of functions - Uniform

Analogous to uniform continuity (which of course we have provided yet), n_ε is not a function of *x*: $\forall n_\varepsilon \neq n_\varepsilon(t) \ni \|f_n(t) - f(t)\| < \varepsilon \;\; \forall n > n_\varepsilon$, and all $\forall t \in \mathcal{X}$. If there is a defined metric $\mathrm{d}(f, g)$ then we can say $f_n \xrightarrow{u} f \Leftrightarrow \mathrm{d}(f_n, f) \to 0$.

In a NLS such as R^n, one such metric is the norm-induced metric, or the supremum metric. Since we will normally be concerned with distribution functions, suppose $f, g : R^d \to R$; define

$$\mathrm{d}_\infty(f,g) \triangleq \|f,g\|_\infty = \sup_{t \in R^d} \|f(t) - g(t)\|_{p=1} \overset{p=1}{=} \sup_{t \in R^d} |f(t) - g(t)| .$$

Continuity

Heine's definition/concept is that a continuous function maps convergent sequences into convergent sequences

Continuity of a function – pointwise

Continuity of a function - uniform

Indented stuff

- Bulleted stuff

Real Analysis

Fundamental Theorems

FTA

Fundamental Theorems of Calculus[3]

Mean Value Theorem (MVT):

Suppose g is continuous on *[a,b]*, and differentiable on *(a,b)*. Then $\exists$ (at least one)

[3] data\equation\Handy.Math.wmf

$\lambda \in (a,b)$ such that $g'(\lambda) = \dfrac{g(b)-g(a)}{b-a} \Leftrightarrow g(b)-g(a) = g'(\lambda)(b-a)$.

Leibnitz Rule: for differentiable $f(x,\theta)$, $a(\theta)$ and $b(\theta)$:

$$\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(x,\theta)dx = f(b(\theta),\theta)\frac{d}{d\theta}b(\theta) - f(a(\theta),\theta)\frac{d}{d\theta}a(\theta) + \int_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial\theta} f(x,\theta)dx$$

FTS

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FTP

References

1. Dobelman, J.A., *Standard Regularity Conditions for Statistics*, Rice University (2011)