**Hypothesis Testing and Interval Estimation.** With a test of hypothesis we get all the distribution information from the Null Hypothesis, and then determine the "rejection region" for the test statistic based on the test’s significance level $\alpha$ (say 5%). Then if the value we get for our statistic is so outrageous that it falls in the reject region, we say the parameters specified in the null must be rejected.

**Relationship to (1-$\alpha$)% Confidence Interval.** The relationship between a 2-sided significance test and the $(1 - \alpha)$% confidence interval is a dual. A level-$\alpha$ significance test rejects $H_0 : \mu = \mu_0$ exactly when $\mu_0$ falls outside a $(1 - \alpha)$% confidence interval for $\mu$.

**Basic Hypothesis Testing and Power of a Test.** We provide the basics of hypothesis testing and then calculate the power of a test with regard to a specific alternative hypothesis. We first define our sample data as $X = x_1, x_2, \ldots, x_n$. We define a “critical region” $W$, or rejection region, of “size” (probability), say $\alpha = .05$. We then say $P(X$ is in the reject region given null hypothesis) $\leq \alpha = .05$, or $P(X \in W \mid H_0) \leq \alpha$ which means we reject the null with no more than 5% chance if its true. Let us say $\alpha$ exactly=.05. Our procedure is:

a. **State the Hypothesis.** $H_0: \mu = \mu_0$ (for example) $H_a: \mu > \mu_0$ (1-sided test)

b. **State Size of Test $\alpha$.** This is the significance level, say $\alpha = .05$

c. **Devise a statistic and its distribution under $H_0$:** e.g., for $X \sim N(\mu, \sigma^2)$, say $T(x) = \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$, and, assuming $\sigma$ known, we then have that $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$, with standard deviation is $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$.

d. **Draw the Critical Region on the Sampling Distribution specified by the Null:**

For $\alpha = .05$, reject if $z_\alpha \geq 1.96$ (from the Normal table for $p=.05$), or $\bar{X} \geq \bar{X}_0 = \sigma_{\bar{X}} z_\alpha + \mu_0$

e. **Calculate statistic, see if you reject or not.** Example: (problem 6.63), SRS $n=500$

$H_0: \mu = 450 \quad \alpha = .01$ gives critical value $z_\alpha = 2.326$.

$H_a: \mu > 450 \quad$ Since $\sigma_{\bar{X}} = 100/\sqrt{500} = 4.4721$, our critical value in terms of $\bar{X}$ is for $\bar{X} \geq \bar{X}_0 = 4.4721(2.326)+450 = 460.4$, so we would reject $H_0$ if our $\bar{X}$ turns out to be greater than 460.4.
f. **Power of the Test.** The power ($\beta$) of the test is the probability of rejecting $H_0$ when it is false, i.e., $P(\text{reject } H_0 \mid H_a)$. We want this probability to be as large as possible, to have the highest power possible for all parameter values in the alternative hypothesis. $P(X \in W \mid H_a) = P(\bar{X} \geq \bar{X}_0 \mid H_a) = P(\bar{X} \geq 460.4 \mid H_a)$, so we have to compute this probability in terms of $H_a$. We can actually calculate the power as a function of all parameter values in the alternative parameter space and plot the power function. In this example we only plot for certain values of the parameter.

Under a specific value for $\mu_i$ in $H_a$, $\mu = \frac{\bar{X} - \mu_i}{\sigma_{\bar{X}}}$. Doing some subtraction and division inside the probability statement gives $P(\bar{X} \geq \bar{X}_0 \mid H_a) = P(\frac{\bar{X} - \mu}{\sigma_{\bar{X}}} \geq \frac{\bar{X}_0 - \mu_i}{\sigma_{\bar{X}}})$, but $\bar{X} - \text{"true" mean} = z$, so we obtain $P(z \geq \frac{\bar{X}_0 - \mu_i}{\sigma_{\bar{X}}}) = \beta$, the power of the test for $\mu_i$, a specific value in the alternative hypothesis.

**Example (Problem 6.63, CONT’D):** We check for $\mu_i$ in $H_a = 460$.

$\beta = P(z \geq \frac{\bar{X}_0 - \mu_i}{\sigma_{\bar{X}}}) = P(z \geq \frac{460.4 - 460}{4.4721}) = P(z \geq 0.894) = .4644$

Note: We get this from either the SECOND Z table (1-Positive z values), or by symmetry in the FIRST z table for $z = -.0894$.

Since we prefer $\beta > .8$ for sensitivity, this test is pretty lame for detecting a 10 point difference. Suppose however that $\mu_i = 475$, a 25 point difference. Here, $\beta_{475} = P(z \geq \frac{460.4 - 475}{4.4721}) = P(z \geq 3.26) = .9994$, practically the whole normal curve to the right, and we still have $P(\text{reject } H_0 \text{ falsely})$ at only $\alpha = .05$.

**For a 2-Sided Test,** the same procedure applies. The difference is that the critical region is the union of 2 disjoint sets. For symmetric distributions such as Normal and the “t”, we determine $P(X \in W) = \alpha$ as

$P(X \in W) = P(X \in W_{LEFT} \ or \ X \in W_{RIGHT}) = P(X \in W_{LEFT}) + P(X \in W_{RIGHT}) = \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$

We then draw our critical region(s) and determine the values of the statistic for which we will reject $H_0$. Then power can be calculated as before but with two regions of interest.
Another Complete Example (problem 6.64). Mean amount of Coke in bottle. SRS, n=1 sixpack=6 bottles. Assume X is content in bottles is Normal(\(\mu, \sigma = 3\)). Devise a Hypothesis Test on actual contents mean \(\mu\).

a. \(H_0: \mu = 300\)  
   \(H_a: \mu < 300\)

b. \(\alpha = .05\) gives 1-sided critical value \(z_\alpha = -1.645\) (conveniently given)

c. Statistic. \(\bar{X}\) under \(H_0\) is \(N(\mu_0, \sigma_\pi) = N(300, \frac{3}{\sqrt{6}}) = N(300, 1.225)\)

d. Critical Region. Reject if \(\bar{X} \leq \bar{X}_0 = \sigma_\pi z_\alpha + \mu_0 = 1.225(-1.645) + 300 = 297.98\). Repeating, reject if \(\bar{X} \leq \bar{X}_0 = 297.98\).

e. Compute Power, ability of the test to reject \(H_0\) if it’s false.

<table>
<thead>
<tr>
<th>(\mu_1)</th>
<th>(\beta = P(z \leq \frac{X_0 - \mu_1}{\sigma_\pi}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>299</td>
<td>(P(z \leq \frac{297.98 - 299}{1.225}) = P(z \leq -0.832) \approx .202)</td>
</tr>
<tr>
<td>295</td>
<td>(P(z \leq \frac{297.98 - 295}{1.225}) = P(z \leq 2.433) \approx .9926)</td>
</tr>
</tbody>
</table>

Again, for \(\mu_1\) close to \(\mu_0\) (299 vs. 300), the power is low, but when \(\mu_1\) vs. \(\mu_0\) is (295 vs. 300), a difference of only 1.6%, the power is almost 1!