Frequently in Probability and Statistics we need to calculate or estimate integrals and expectations; usually this is done through limiting arguments in which a sequence of integrals is shown to converge to the one whose value we need. Here are some important properties of integrals for any measurable set $\Lambda \in \mathfrak{F}$ and random variables $\{X_n\}$, X, Y, useful for bounding or estimating the integral of a random variable X

1. $\int_\Lambda X\,d\mathsf{P}$ is well-defined and finite if and only if $\int_\Lambda |X|\,d\mathsf{P} < \infty$, and $\left|\int_\Lambda X\,d\mathsf{P}\right| \le \int_\Lambda |X|\,d\mathsf{P}$. We can also define $\int_\Lambda X\,d\mathsf{P} \le \infty$ for any X bounded below by some $b > -\infty$.

2. **Lebesgue's Monotone Convergence Thm:** If $0 \le X_n \nearrow X$, then $\int_\Lambda X_n\,d\mathsf{P} \nearrow \int_\Lambda X\,d\mathsf{P} \le \infty$. In particular, the sequence of integrals converges (possibly to $+\infty$).

3. **Lebesgue's Dominated Convergence Thm:** If $X_n \to X$, and if $|X_n| \le Y$ for some RV $Y \ge 0$ with $\mathsf{E}Y < \infty$, then $\int_\Lambda X_n\,d\mathsf{P} \to \int_\Lambda X\,d\mathsf{P}$ and $\int_\Lambda |X|\,d\mathsf{P} \le \int_\Lambda Y\,d\mathsf{P} < \infty$. In particular, the sequence of integrals converges to a finite limit.

4. **Fatou's Lemma:** If $X_n \ge 0$ on Λ, then

$$\int_\Lambda (\liminf X_n)\,d\mathsf{P} \le \liminf \Big(\int_\Lambda X_n\,d\mathsf{P} \Big).$$

   The two sides may be unequal (example?), and the result is false for lim sup. Is "$X_n \ge 0$" necessary?

5. **Fubini's Thm:** If *either* each $X_n \ge 0$, *or* $\sum_n \int_\Lambda |X_n|\,d\mathsf{P} < \infty$, then the order of integration and summation can be exchanged: $\sum_n \int_\Lambda X_n\,d\mathsf{P} = \int_\Lambda \sum_n X_n\,d\mathsf{P}$. If both these conditions fail, the orders may not be exchangeable (example?)

6. For any $p > 0$, $\mathsf{E}|X|^p = \int_0^\infty p\,x^{p-1}\mathsf{P}[|X| > x]\,dx$ and $\mathsf{E}|X|^p < \infty \Leftrightarrow \sum_{n=1}^\infty n^{p-1}\mathsf{P}[|X| \ge n] < \infty$. The case $p = 1$ is easiest and most important: if $S \equiv \sum_{n=1}^\infty \mathsf{P}[|X| \ge n] < \infty$, then $S \le \mathsf{E}|X| < S+1$. If X takes on only nonnegative integer values, $\mathsf{E}X = S$.

7. If μ_X is the distribution of X, and if f is a measurable real-valued function on $\mathbb{R}$, then $\mathsf{E}f(X) = \int_\Omega f(X(\omega))\,d\mathsf{P} = \int_{\mathbb{R}} f(x)\,\mu_X(dx)$ if either side exists. In particular, $\mu = \mathsf{E}X = \int x\,\mu_X(dx)$ and $\sigma^2 = \mathsf{E}(X-\mu)^2 = \int (x-\mu)^2\,\mu_X(dx)$.

8. **Hölder's Inequality:** Let $p > 1$ and $q = \frac{p}{p-1}$ (*e.g.*, $p = q = 2$ or $p = 1.01$, $q = 101$). Then $\mathsf{E}\,XY \le \mathsf{E}\,|XY| \le \left[\mathsf{E}|X|^p\right]^{\frac{1}{p}}\left[\mathsf{E}|Y|^q\right]^{\frac{1}{q}}$. In particular, for $p = q = 2$,
Cauchy-Schwartz Inequality: $\mathsf{E}\,XY \le \mathsf{E}\,|XY| \le \sqrt{\mathsf{E}X^2\,\mathsf{E}Y^2}$.

9. **Minkowski's Inequality:** Let $1 \le p \le \infty$ and let $X, Y \in L_p(\Omega, \mathcal{F}, \mathsf{P})$. Then
$$(\mathsf{E}|X+Y|^p)^{\frac{1}{p}} \le (\mathsf{E}|X|^p)^{\frac{1}{p}} + (\mathsf{E}|Y|^p)^{\frac{1}{p}}$$
so the norm $\|X\|_p \equiv (\mathsf{E}|X|^p)^{\frac{1}{p}}$ obeys the triangle inequality on $L_p(\Omega, \mathcal{F}, \mathsf{P})$ (what if $p < 1$?).

10. **Jensen's Inequality:** Let $\varphi(x)$ be a convex function on $\mathbb{R}$, X an integrable RV. Then $\varphi(\mathsf{E}[X]) \le \mathsf{E}[\varphi(X)]$. Examples: $\varphi(x) = |x|^p$, $p \ge 1$; $\varphi(x) = e^x$; $\varphi(x) = [0 \vee x]$. The equality is *strict* if $\varphi(\cdot)$ is strictly convex and X has a non-degenerate distribution.

11. **Markov's & Chebychev's Inequalities:** If φ is positive and increasing, then $\mathsf{P}[|X| \ge u] \le \mathsf{E}[\varphi(|X|)]/\varphi(u)$. In particular $\mathsf{P}[|X-\mu| > u] \le \frac{\sigma^2}{u^2}$ and $\mathsf{P}[|X| > u] \le \frac{\sigma^2+\mu^2}{u^2}$.

12. **One-Sided Version:** $\mathsf{P}[X > u] \le \frac{\sigma^2}{\sigma^2+(u-\mu)^2}$
(pf: $\mathsf{P}[(X-\mu+t) > (u-\mu+t)] \le ?$)

13. **Hoeffding's Inequality:** If $\{X_j\}$ are independent and $(\exists\ \{a_j, b_j\})$ s.t. $\mathsf{P}[a_j \le X_j \le b_j] = 1$, then $(\forall c > 0)$, $S_n := \sum_{j=1}^n X_j$ satisfies $\mathsf{P}[S_n - \mathsf{E}S_n \ge c] \le \exp\left(-2c^2/\sum_1^n |b_j - a_j|^2\right)$. Hoeffding proved this improvement on Chebychev's inequality (at UNC) in 1963. See also related **Azuma**'s inequality (1967), **Bernstein**'s inequality (1937), and **Chernoff** bounds (1952).