MEASURE THEORY COURSE NOTES

LANCE MILLER

ABSTRACT. This is a copy of the course notes for Dr. Stu Sidney's Math 303 Measure and Integration course offered in Spring 2005 at the University of Connecticut Included are homework assignments and solutions, as well as the exams. In addition I have referenced Dr. Richard Bass's course notes from the same course, and provided the preparation material used for the Measure theory Qualifying exam a the University of Connecticut including solutions to many old exam problems. Many of the solutions came from group discussions of the graduate students preparing, as well as the wonderful compilation of solutions to old qualifying exams

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[1]These notes are severely unedited. Anyone whom would like to take on the project of cleaning these notes up is more than welcome

[2]The collaboration of solutions to old exams was carried out primarily by Dr. Molli Jones, and these notes are truly a revision of hers

Contents

1. A quick review of Riemann Integration

The Riemann integral has the following equivalent definitions:

Definition 1.1. *Let f be a real valued function on $[a,b]$. Then we define for each partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$, the numbers*

$$M_k = \sup_{x\in[x_{k-1},x_k]} f(x) \qquad m_k = \inf_{x\in[x_{k-1},x_k]} f(x)$$

Now we define $L(f,P) = \sum_{k=1}^{n} m_k \triangle x_k$ and $U(f,P) = \sum_{k=1}^{n} M_k \triangle x_k$. Then we have that $L\int_a^b f(x)dx = \sup_P L(f,P)$ and $U\int_a^b f(x)dx = \inf_P U(f,P)$ where the $\sup$ *and* $\inf$ *are taken over all partitions P. If for a given function $L\int_a^b f(x) = dx = U\int_a^b f(x)dx$ then we say that f is Riemann integral, and $\int_a^b f(x)dx$ is the common value*

Definition 1.2. *A function f on $[a,b]$ provided $\forall\lambda > 0, \mu > 0$ we have that there is a partition P such that $\sum_{\{k|M_k-m_k>\lambda\}} \triangle x_k < \mu$.*

Though the Riemann integral enjoys some very satisfactory properties such linearity and being closed under uniform limits it has many shortcommings, most notably is that it is not closed under pointwise convergence.

Example 1.3. *Consider the sequence*

$$f_m(x) = \lim_{n\to\infty} \cos^{2n}(m!x\pi)$$

. When $m!x \in \mathbb{Z}$, then $f_m(x) = 1$, and for any other $x, f_m(x) = 0$. Let

$$f(x) = \lim_{m\to\infty} f_m(x)$$

. Then for any $x \in \mathbb{R}\setminus\mathbb{Q}$, we have $f_m(x) = 0\forall m$ thus $f_m(x) = 0$. If $x \in \mathbb{Q}$, let $x = \frac{p}{q}$. If $m \geq q$, then $m!x \in \mathbb{Z}$ and so $f_m(x) = 1\forall m \geq q$ which gives that $f(x) = 1$. Thus

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

Which is not Riemann integrable.

Example 1.4. *Let $R = \mathbb{Q}\cap[0,1]$ and let $\{r_0, r_1, r_2, \ldots\}$ be an enumeration. Then let $f_n = \chi(\{r_0,\ldots,r_n\})$. Then we have that $f_n \to \chi(\mathbb{Q}\cap[0,1])$ pointwise and $\chi(\mathbb{Q}\cap[0,1])$ is not Riemann integrable.*

2. Algebras of Sets and σ-Algebras

The study of measure and integration begins with the notion of algebras and σ-algebras. These are families of sets which are closed under certain set theoretic operations. This is an anaologue to the generalization of open sets in topology. The connection between measureable sets and functions and open sets and continuous functions will continue later on.

Definition 2.1. *Given a set X, then a family $\mathcal{F} \subset \wp(X)$, then $\mathcal{F}$ is called a (boolean) algebra provided*

(1) $\emptyset \in \mathcal{F}$
(2) $A \cup B \in \mathcal{F}$, $\forall A, B \in \mathcal{F}$
(3) $A^c \in \mathcal{F}$, $\forall A \in \mathcal{F}$

We can extend this definition slightly as follows:

Definition 2.2. *Given a set X, then a family $\mathcal{F} \subset \wp(X)$, then $\mathcal{F}$ is called a σ-algebra provided*

(1) $\emptyset \in \mathcal{F}$
(2) $\bigcup A_i \in \mathcal{F}$, $\forall \{A_i\} \subset \mathcal{F}$
(3) $A^c \in \mathcal{F}$, $\forall A \in \mathcal{F}$

By far the most important propositions involving algebras and $\sigma-$algebras is the following:

Proposition 2.3. *Given any set X and any collection $\mathcal{C} \subset \wp(X)$. Then there exists $\sigma(\mathcal{C})$ ($alg(\mathcal{C})$) a σ-algebra (algebra) containing $\mathcal{C}$ such that if $\mathcal{F} \subset \wp(X)$ is any other $\sigma-$algebra (algebra) containing $\mathcal{C}$ then, $\sigma(\mathcal{C}) \subset \mathcal{F}$ ($alg(\mathcal{C}) \subset \mathcal{F}$).*

Proof. This is so as arbitrary intersections of $\sigma-$algebras (algebras) are also $\sigma-$algebras (algebras). Thus we can given the collection $\mathcal{C}$ let $S = \{\mathcal{F} \subset \wp(X) | \mathcal{C} \subset \mathcal{F}$ and $\mathcal{F}$ is a $\sigma-$algebra (algebra) $\}$. This collection is non-empty as $\wp(X) \in S$. So we have that $\sigma(\mathcal{C}) = \bigcap_{\{\mathcal{F} \in S\}} \mathcal{F}$ is the required $\sigma-$algebra (algebra). //

2.1. **Borel, F_σ and G_δ sets.** The above proposition is classically non-constructive. Given a collection $\mathcal{C}$ we can form $alg(\mathcal{C})$ as follows. Let $\mathcal{C}_0 = \mathcal{C}$. Then let $\mathcal{C}_1 = \mathcal{C}_0 \cup \mathcal{C}^c$ consist of members of $\mathcal{C}$ or their complement. That is $\mathcal{C}_1 = \{A^* | A \in \mathcal{C}$ or $A^c \in \mathcal{C}\}$. We can then form $\mathcal{C}_2 = \{\bigcap_{i=1}^n A_i | A_i \in \mathcal{C}_1\}$, and finally $\mathcal{C}_3 = \{\bigcup_{i=1}^n | A_i \in \mathcal{C}_2\}$. One can check that $\mathcal{C}_3 = alg(\mathcal{C})$. However if one tries to repeat one quickly realizes that it can go on ad infinitum. With the collection of all the open subsets in a topological space in mind we quickly come to the following definitions:

Definition 2.4. *A set is a F_σ set provided it is a countable union of closed sets.*

Definition 2.5. *A set is a G_δ set provided it is a countable intersection of open sets*

These sets arise naturally when one tries to find the smallest $\sigma-$algebra generated by all the open (closed)sets. The sets in this collection are called Borel.

We pause to give some examples of $\sigma-$algebras.

Example 2.6. *Naturally the powerset $\wp(X)$ of X is a $\sigma-$algebra.*

Example 2.7. *Given X a set. Let $\mathcal{C} = \{A \in \wp(X) || A| = \omega$ or $|A^c| = \omega\}$.*

3. Measures

Now that we have $\sigma-$algebras, we will need a way to describe their size in order to integrate over their elements.

Definition 3.1. *Let $\mathcal{F}$ be a $\sigma-$algebra. Then a function $\mu : \mathcal{F} \to [0, \infty]$ is called a (positive) measure provided:*

(1) $\mu(\emptyset) = 0$
(2) $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$ *for pairwise disjoint collections* $\{A_i\} \subset \mathcal{F}$

We will called a triple $(X, \mathcal{F}, \mu)$ consisting of a $\sigma-$algebra contained in $\wp(X)$ and a measure on $\mathcal{F}$ a measure space. Such a space will be called finite if $\mu(X) < \infty$ and $\sigma-$finite provided $X = \bigcup_i X_i$ where $\mu(X_i) < \infty$. A measure space is complete provided that for any set $A \in \mathcal{F}$ such that $\mu(A) = 0$ then $\forall B \subset A$ we have that $B \in \mathcal{F}$. We will see soon that $\mu(B)$ must be zero as well.

Example 3.2. *(Counting measure) Let X be a non-empty set. Then $(X, \wp(X), \mu)$ where $\mu(A) = |A|$ for $|A| < \infty$, and $\mu(A) = \infty$ for $|A| \geq \omega$.*

Example 3.3. *Given X a set. Let $\mathcal{C} = \{A \in \wp(X) | |A| = \omega$ or $|A^c| = \omega\}$. Then $\mu(A) = 1$ for $|A^c| = \omega$ and $\mu(A) = 0$ for $|A| = \omega$ is a measure.*

We now pause to give some lemmas needed to compute with measures. For each of the following fix a measure space $(X, \mathcal{F}, \mu)$.

Lemma 3.4. *For $A \subset B$ where $A, B \in \mathcal{F}$, then $\mu(A) \leq \mu(B)$.*

Proof. Write $B = A \cup (B \setminus A)$. Then $\mu(B) = \mu(A) + \mu(B \setminus (A))$. //

Lemma 3.5. *For any collection $\{A_i\} \subset \mathcal{F}$. Then $\mu(\bigcup_i A_i) \leq \sum_i \mu(A_i)$.*

Proof. Let $B_1 = A_1$, $B_2 = A_2 \setminus B_1$, $B_3 = A_3 \setminus (B_1 \cup B_2), \ldots$. Then the collection $\{B_i\}$ are disjoint and $\bigcup_i B_i = \bigcup_i A_i$. Then we have that $\mu(\bigcup_i A_i) = \mu(\bigcup_i B_i) = \sum_i \mu(B_i) \leq \mu(A_i)$, the last inequality holds from the previous lemma. //

Lemma 3.6. *Given a collection $A_1 \supset A_2 \supset \ldots$ all in $\mathcal{F}$, such that $\mu(A_1) < \infty$, then $lim\mu(A_i) = \mu(\bigcap A_i)$.*

Proof. Let $A = \bigcap A_i$ and $B_i = A_i \setminus A_{i+1}$. Then we have that $\{B_i\}$ are pairwise disjoint and so $\mu(A_1 \setminus A) = \mu(\bigcup_{\{i \geq 1\}} B_i) = \sum \mu(B_i) = \sum \mu(A_i \setminus A_{i+1})$. Then we have that $\mu(A_1) = \mu(A) + \mu(A_1 \setminus A)$ and $\mu(A_i) = \mu(A_{i+1}) + \mu(A_i \setminus A_{i+1})$. And we have that $\mu(A_1) - \mu(A) = \sum \mu(A_i) - \mu(A_{i+1}) = \lim \sum_{i=1}^{n} \mu(A_i) - \mu(A_{i+1}) = \lim \mu(A_1) - \mu(A_n) = \mu(A_1) - lim\mu(A_n)$. Now since $\mu(A_1) < \infty$ we can subtract to get the result. //

This last lemma is probably the most useful.

4. Construction of Lebesgue Measure on $\mathbb{R}$

We will now go through the construction of the lebesgue measure on $\mathbb{R}$.

Given a set $E \subset \mathbb{R}$, we can define $m^*(E) = \inf_{\{E \subset \bigcup I_n\}} \sum l(I_n)$ where $\{I_n\}$ is a collection of intervals. We have the following trivial results:

Lemma 4.1. *For m^* defined above, we have the following*

1. *For $A \subset B$, then $m^*(A) \subset m^*(B)$*
2. *$m^*(\bigcup A_i) \leq \sum m^*(A_i)$*
3. *If E is countable then $m^*(E) = 0$*
4. *$m^*(A + r) = m^*(A)$.*

Proof. For (1) this is clear as any cover of B is also a cover of A. (3) follows from (2) and the fact that $m^*(\{r\}) = 0, \forall r \in \mathbb{R}$. (4) is trivial, so it suffices to show (2). Without loss of generality $m^*(A_i) < \infty$. Let $\epsilon > 0$, and for each i choose $\{I_{n,i}\}$ such that $A_i \subset \bigcup_n I_{(n,i)}$ and $\sum_n l(I_{(n,i)}) = m^*(A_i) + \frac{\epsilon}{2^i}$. Then we have that $m^*(\bigcup A_i) \leq \sum_{(n,i)} l(I_{(n,i)}) \leq \sum m^*(A_i) + \frac{\epsilon}{2^i} = \sum m^*(A_i) + \epsilon$, which gives the result. //

We now define what it means for a subset of the real numbers to be measureable.

Definition 4.2. *Let E be a subset of real numbers. Then E is measureable provided $\forall A$ we have $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$.*

Ideally we want to form a $\sigma-$algebra. We have the following theorem:

Proposition 4.3. *The collection* $\mathcal{M} = \{E \subset \wp(\mathbb{R}) | E$ *is measurable* $\}$ *is a* $\sigma-$*algebra.*

Proof. $\emptyset \in \mathcal{M}$ as $m*(A) = m*(A) + 0$. Likewise if $E \in \mathcal{F}$, then $E^c \in \mathcal{M}$ also by defintion. So it suffices to show that $\bigcup_i E_i \in \mathcal{M}$ for a countable collection $\{E_i\} \subset \mathcal{M}$. First we show that $E_1 \cup E_2$ for any $E_1, E_2 \in \mathcal{M}$. Let A be any set. Then we have that $m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)$. Further since $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2 \cap E_1^c)$ we have that $m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap E_1^c \cap E_2^c) \leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c) + m^*(A \cap E_1^c \cap E_2^c) = m^*(A \cap E_1) + m^*(A \cap E_1^c) = m^*(A)$. Induction gives finite sums of memebers of $\mathcal{M}$ are in $\mathcal{M}$. Given any collection $\{E_i\} \subset \mathcal{M}$. Then we can construct $B_n = E_n \setminus (E_1 \cup \ldots \cup E_{n-1}) = E_n \cap E_1^c \cap \ldots \cap E_{n-1}^c$, then $\{B_i\} \subset \mathcal{M}$, $\bigcup B_i = \bigcup E_i$ and $\{B_i\}$ are pairwise disjoint. Now let $C_n = \bigcup_{i=1}^n B_i$, and $C = \bigcup B_i$. Then $m^*(A \cap C_n) = m^*(A \cap C_n \cap B_n) + m^*(A \cap C_n \cap B_n^c) = m^*(A \cap B_n) + m^*(A \cap C_{n-1})$, repeating this same argument for $m^*(A \cap C_{n-1}), m^*(A \cap C_{n-2}), \ldots$ we have $m^*(A \cap C_n) = \sum_{i=1}^n m^*(A \cap B_i)$. Thus $m^*(A) = m^*(A \cap C_n) + m^*(A \cap C_n^c) \geq \sum_{i=1}^n m^*(A \cap B_i) + m^*(A \cap C^c)$. Now taking the limit as $n \to \infty$ we obtain $m^*(A) \geq \sum m^*(A \cap B_i) + m^*(A \cap C^c) \geq m^*(\bigcup A \cap B_i) + m^*(A \cap C^c) = m^*(A \cap C) + m^*(A \cap C^c)$. This gives that $C = \bigcup B_i = \bigcup E_i \in \mathcal{M}$. //

Now we will demonstrate some measureable sets.

Lemma 4.4. *Let* E *be a set such that* $m^*(E) = 0$*, then* E *is measurable.*

Proof. Let $m^*(E) = 0$. Let $A \in \wp(\mathbb{R})$. Then $A \cap E \subset E$ so $m^*(A \cap E) = 0$. LIkewise $m^*(A \cap E^c) = 0$, therefore we have that $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$, therefore E is measurable. //

Corollary 4.5. *Any countable set is measurable*

Lemma 4.6. *Any interval is measurable.*

Proof. Since the measurable sets are a $\sigma-$algebra, it suffices to show that (a, ∞) is measurable for any $a \in \mathbb{R}$. Let $A \in \wp(\mathbb{R})$. Now let $A_1 = A \cap (a, \infty)$ and $A_2 \cap (-\infty, a]$. Then we must show that $m^*(A_1) + m^*(A_2) \leq m^*(A)$. Likewise we are done if $m^*(A) = \infty$, so let $m^*(A) < \infty$. Now, let $\epsilon > 0$. Then we can find $\{I_n\}$, $A \subset \bigcup I_n$ and $\sum l(I_n) \leq m^*(A) + \epsilon$. Let $I_n' = I_n \cap (a, \infty)$ and $I_n'' = I_n \cap (-\infty, a]$. Then $l(I_n) = l(I_n') + l(I_n'')$. Likewise $A_1 \subset \bigcup I_n'$ so $m^*(A_1) \leq m^*(\bigcup I_n') \leq \sum m^*(I_n')$. Similarly $m^*(A_2) \leq \sum m^*(I_n'')$. $m^*(A_1) + m^*(A_2) \leq \sum(m^*(I_n') + m^*(I_n'')) \leq \sum l(I_n) \leq m^*(A) + \epsilon$. This gives that $m^*(A_1) + m^*(A_2) \leq m^*(A)$, thus (a, ∞) is measurable. //

This gives that $\mathcal{M}$ is a $\sigma-$algebra which contains the open sets of $\mathbb{R}$, and hence any borel set. Now if we restrict $m = m^*|_{\mathcal{M}}$ then we obtain a measure.

Proposition 4.7. *The triple* $(\mathbb{R}, \mathcal{M}, m = m^*|_{\mathcal{M}})$ *is a measure space.*

Proof. So far we have seen that $\mathcal{M}$ is a $\sigma-$algebra. Likewise we have that $m^*(\emptyset) = 0$ so it suffices to show that $m^*(\bigcup E_i) = \sum m^*(E_i)$ for $\{E_i\} \subset \mathcal{M}$. Likewise we have that $m^*(\bigcup E_i) \leq \sum m^*(E_i)$. First we show that $m^*(A \cup B) = m^*(A) + m^*(B)$ for $A, B \in \mathcal{M}$ disjoint. This is so as $\mathbb{R} \cap (A \cup B)$ is a measurable set, thus we have $m^*(A \cup B) = m^*(\mathbb{R} \cap (A \cup B)) = m^*(\mathbb{R} \cap (A \cup B) \cap A) + m^*(\mathbb{R} \cap (A \cup B) \cap A^c) = m^*(\mathbb{R} \cap A) + m^*(\mathbb{R} \cap B) = m^*(A) + m^*(B)$. Induction give the result for finite sums. Now $\bigcup E_i \supset \bigcup_{i=1}^n E_i$ which gives that $m^*(\bigcup E_i) \geq m^*(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n m^*(E_i)$.

This gives that $m^*(\bigcup E_i) \geq \sum m^*(E_i)$ which gives that $m^*(\bigcup E_i) = \sum m^*(E_i)$ and so $m = m^*|_{\mathcal{M}}$ is a measure. //

This gives that $(\mathbb{R}, \mathcal{M}, m)$ is a $\sigma-$finite complete measure space. We will call the measure given the lebesgue measure on $\mathbb{R}$. Now it is an unfortunate truth that there are sets which are not measurable.

Proposition 4.8. *Given any set $E \in \mathcal{M}$ such that $m(E) > 0$, then there is a set $A \subset E$ such that $A \notin \mathcal{M}$.*

Proof. Define $x\ y$ provided $x - y \in \mathbb{Q}$. Then this is an equivalence relation on $[0, 1]$. Using the axiom of choice let A be the set formed by choosing a element from each equivalence class. Now using translation invarience we have that $m^*(A+q) = m^*(A)$. Moreover these sets are disjoint for different rationals q. Now $[0, 1] \subset \bigcup_{q\in[-2,2]\cap\mathbb{Q}}(A+q)$ so $m^*(A) > 0$. However $\sum_{q\in[-2,2]\cap\mathbb{Q}} m^*(A+q)$ must be either 0 or infinte as $\{A + q\}$ are disjoint and non-negative. Further we have that $\bigcup_{q\in[-2,2]\cap\mathbb{Q}}(A+q) \subset [-6, 6]$ so this gives that $m^*(A) = 0$ which is a contradiction, therefore A is not measurable. //

5. Measureable Functions

The analogy between measure theory and topology is picked up again here. As $\sigma-$algebras are to topologies, measureable functions are to continuous functions. Recall that a continuous function can be defined as one whoes inverse images of open sets are open. Likewise one can define measurable functions as those whoes inverse images of open sets are measurable. We formalize this as following the next result.

Proposition 5.1. *For a given measure space $(X, \mathcal{F}, \mu)$ and a function $f : X \to \mathbb{R}$, then the following are equivalent:*

1. $\{x|f(x) > a\} \in \mathcal{F}$, $\forall a \in \mathbb{R}$
2. $\{x|f(x) \geq a\} \in \mathcal{F}$, $\forall a \in \mathbb{R}$
3. $\{x|f(x) \leq a\} \in \mathcal{F}$, $\forall a \in \mathbb{R}$
4. $\{x|f(x) < a\} \in \mathcal{F}$, $\forall a \in \mathbb{R}$

Proof. We have by complementation $1 \Leftrightarrow 3$ and $2 \Leftrightarrow 4$, so it suffices to show $1 \Leftrightarrow 2$. Now we have that $1 \Rightarrow 2$ as $f^{-1}([a, \infty)) = \bigcap f^{-1}((a - \frac{1}{n}, \infty))$. Likewise $2 \Rightarrow 1$ as $f^{-1}((a, \infty)) = \bigcup f^{-1}([a + \frac{1}{n}, \infty))$. //

Given this we have the following definition:

Definition 5.2. *A function $f : X \to \mathbb{R}$ where $(X, \mathcal{F}, \mu)$ is a measure space is measurable provided one (equivalently all) of the conditions in the previous proposition hold.*

Corollary 5.3. *Monotonic and continuous functions are all lebesgue measurable.*

It will be useful in the future to distinquish the following measurable functions:

Definition 5.4. *Given a space $(X, \mathcal{F}, \mu)$ and a measureable function f the set $supp(f) = \{x|f(x) \neq 0\}$ is called the support of f. A measurable function is finitely supported provided $\mu(supp(f)) < \infty$.*

We now give some tools to compute measurable functions:

Proposition 5.5. *Given a measure space $(X, \mathcal{F}, \mu)$ and measurable functions f, g and a set of measurable functions $\{f_n\}$, then $f+g, fg, \sup\{f_n\}, \inf\{f_n\}$ are all measurable*

Proof. First we we show that $f+g$ is measurable. Consider $\{x|f(x)+g(x)<a\}$ for $a \in \mathbb{R}$. This is the same as $\{x|f(x)<a-g(x)\} = \bigcup_{q\in\mathbb{Q}}(\{x|f(x)<q\} \cap \{x|g(x) < a-q\}$. This gives that $f+g$ is measurable. For fg, we first note that this is trivial if f is constant. Futher we have that if $a \geq 0$ then $\{x|f^2(x) > a\} = \{x|f(x) > \sqrt{a}\} \cup \{x|f(x) < -\sqrt{a}\}$ is measurable and when $a < 0$ then $\{x|f^2(x) > a\} = X$ is measurable thus f^2 is measurable. Futher since $fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$ we have that fg is measurable. Now let $g = \sup\{f_n\}$. Then $\{x|g(x) > a\} = \bigcup\{x|f_n(x) > a\}$. Dually $\inf\{f_n\}$ is measurable. //

Corollary 5.6. *Given a measure space $(X, \mathcal{F}, \mu)$ and $\{f_n\}$ measurable functions. Then $\overline{\lim} f_n$ and $\underline{\lim} f_n$ are both measurable, hence limits of meaurable functions are measurable.*

Frequently it will be convinent to describe situations that hold except on sets of zero measure. So by convention, given a measure space $(X, \mathcal{F}, \mu)$ a property is said to hold $\mu-$ almost everywhere ($\mu-$a.e.) the set of points on which it doesn't hold has μ measure zero. The following proposition exhibits this type of condition.

Proposition 5.7. *Given a complete measure space $(X, \mathcal{F}, \mu)$, if f is measurable and $f = g$ $\mu-$a.e., then g is measurable.*

Proof. Let $E = \{x|f(x) \neq g(x)\}$. The set $\{x|g(x) < a\} = (\{x|f(x) < a\} \cup \{x \in E|g(x) < a\}) \setminus \{x \in E|g(x) \geq a\}$. Each of the sets on the right hand side are measruable as $(X, \mathcal{F}, \mu)$ is complete.

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Now we will give a method of constructing measurable functions

Lemma 5.8. *Given a space $(X, \mathcal{F}, \mu)$, let D be a dense set of real numbers and $\{B_\alpha\}_{\alpha \in D}$ a collection of measurable sets such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$. Then there is a unique measurable function f such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on B_α^c.*

Proof. Define $f(x) = \inf\{\alpha \in D|x \in B_\alpha\}$. if $x \in B_\alpha$, then $f(x) \leq \alpha$ by definition. Likewise if $x \notin B_\alpha$, then $\forall \beta < \alpha, x \notin B_\beta$ which gives that $\forall \beta < \alpha, f(x) \geq \beta \Rightarrow f(x) \geq \alpha$. Now given $\lambda \in \mathbb{R}$ we have that $\{x|f(x) < \lambda\} = \bigcup_n B_{\alpha_n}$ where $\{\alpha_n\} \to \lambda$. Let g be any other function satisfying the conclusion. Then for $x \in B_\alpha$ we have $g(x) \leq \alpha$ gives $\{\alpha \in D|x \in B_\alpha\} \subset \{\alpha \in D|g(x) \leq \alpha\}$. Likewise for x such that $g(x) < \alpha$ we have that $x \in B_\alpha$ this gives that $\{\alpha \in D|g(x) < \alpha\} \subset \{\alpha \in D|x \in B_\alpha\}$. Further since D is dense we have $g(x) = \inf\{\alpha \in D|\alpha > g(x)\} = \inf\{\alpha \in D|\alpha \geq g(x)\} = \inf\{\alpha \in D|x \in B_\alpha\} = f(x)$. //

Proposition 5.9. *Given a space $(X, \mathcal{F}, \mu)$, let D be a dense set of real numbers and $\{B_\alpha\}_{\alpha \in D}$ such that $\mu(B_\alpha \setminus B_\beta) = 0$ for $\alpha < \beta$. Then there is a measurable function f such that $f \leq \alpha\mu-$a.e. on B_α and $f \geq \alpha\mu-$a.e. on B_α^c. This function is unique in the sense that if g satisfies the same conditions, then $g = f\mu-$a.e.*

Proof. Let C be a countable dense subset of D and let $N = \bigcup B_\alpha \setminus B_\beta$ for $\alpha < \beta$ and $\alpha, \beta \in C$. Then N is a countable union and so is measurable and has measusure zero. Let $B'_\alpha = B_\alpha \cup N$. Then for $\alpha, \beta \in C$ such that $\alpha < \beta$ we have $B'_\alpha \setminus B'_\beta = (B_\alpha \setminus B_\beta) \setminus N = \emptyset$. Thus we have $B'_\alpha \subset B'_\beta$. The previous lemma gives us a measureable function f such that $f \leq \gamma$ on B'_γ and $f \geq \gamma$ on $B^{c'}_\gamma$. Now for each $\alpha \in D$ choose $\{\gamma_n\} \to \alpha$ such that $\alpha > \gamma_n \in C$. Then we have that $B'_\alpha \setminus B'_{\gamma_n} \subset B_\alpha \setminus B_{\gamma_n}$. Thus we have $P = \bigcup B_\alpha \setminus B'_{\gamma_n}$ is measurable and has measure zero. Let $A = \bigcap B'_{\gamma_n}$ Then $f \leq \inf \gamma_n = \alpha$ on A and $B_\alpha \setminus A \subset P$ so $f \leq \alpha \mu-$a.e. on B_α. A similar argument shows $f \geq \alpha\mu-$a.e. on B^c_α. Now let g be a measurable function that satisfies the conclusion. Then $g \leq \gamma\mu-$a.e. on B_γ and $g \geq \gamma\mu-$a.e. on B^c_γ for $\gamma \in C$. This gives that $g \leq \gamma$ on B'_γ and $g \geq \gamma$ on $B^{c'}_\gamma$ except for some measure zero set Q_γ. Let $Q = \bigcup Q_\gamma$. Now $f = g$ except for the set Q which must have measure zero. //

6. Littlewood Principles

The Littlewood theorems basically state the the following:

1. (1) Every measurable set is almost an open set
2. (2) Pointwise convergent sequences of functions are almost uniform
3. (3) Every measurable function is almost continuous

We will make these principles more percise under differing conditions. We should note that Littlewood princples (1) and (3) make reference to topology (open set, continuous function). These theorems can be made in some general settings for a measure space with a nice enough topology. However for these principle we will prove the results for the familar space $(\mathbb{R}, \mathcal{M}, m)$.

Theorem 6.1. *Let $(\mathbb{R}, \mathcal{M}, m)$ be the real numbers with the lebesgue measure. Given E a measurable set., then:*

1. (1) $\forall \epsilon > 0, \exists U$ *open such that* $E \subset U$ *and* $m(U \setminus E) < \epsilon$
2. (2) $\forall \epsilon > 0, \exists C$ *closed such that* $C \subset E$ *and* $m(E \setminus C) < \epsilon$

Proof. Let $E \in \mathcal{M}$. Now let $\epsilon > 0$. The problem is solved if $m(E) = \infty$ as we may take $U = \mathbb{R}$ so assume $m(E) < \infty$. Then since $m(E) = inf_{E \subset \bigcup I_n}(\sum l(I_N))$ we have that there must be $U = \bigcup I_n$ such that $E \subset U$ and $m(E) < m(U) + \epsilon$. This proves the first conclusion. The second is the dual of the first. //

Lemma 6.2. *Let $(X, \mathcal{F}, \mu)$ be a finite measure space and $\{f_n\} \to f - \mu-$a.e. all measurable functions, $\epsilon, \delta > 0$, then $\exists A$ (depending on ϵ), and $N \in \mathbb{N}$ such that $\mu(A) < \delta$ and $\forall n \geq N, |f_n(x) - f(x)| < \epsilon$.*

Proof. Let $\epsilon, \delta > 0$. Let A_1 be the set of $x \in X$ such that $\{f_n(x)\}$ does not converge to f. Then $\mu(A_1) = 0$. Now let $G_k = \{x \notin A_1 | \ |f_k(x) - f(x)| \geq \epsilon\}$. Let $E_n = \bigcup_{k \geq n} G_k$. Then $X \supset E_1 \supset E_2 \supset \ldots$. Further $\bigcap_n E_n = \emptyset$. Since $\mu(X) < \infty$ we have that $\lim \mu(E_n) = \mu(\bigcap E_n) = 0$. Thus we can find N large enough such that $\mu(E_N) < \delta$ and $\forall N \geq n$ we have $|f_n(x) - f(x)| < \epsilon$ by definition. //

Theorem 6.3. *(Egoroff) Let $(X, \mathcal{F}, \mu)$ be a finite measure space and $\{f_n\} \to f\mu-$a.e. all measurable functions. Then $\forall \eta > 0$ there is a set A such that $\mu(A) < \eta$ and $\{f_n(x)\}$ converges uniformly to $f(x)$ on A^c.*

Proof. $\forall k \in \mathbb{N}$ let $\epsilon_k = \frac{1}{k}$ and $\delta_k = 2^{-k}\eta$. Use the previous lemma to choose sets A_k and numbers N_k such that $\mu(A_k) < \delta_k$ and $\forall n \geq N_k$ we have $|f_n(x) - f(x)| < \epsilon_k$ on A^c_k. Let $A = \bigcup A_k$. Then $\mu(A) \leq \sum 2^{-k}\eta = \eta$. Let $\epsilon > 0$. Choose k such that $\frac{1}{k} < \epsilon$, then $\forall x \in A^c \subset A^c_k$ and $n \geq N_k$ we have $|f_n(x) - f(x)| \leq \frac{1}{k} < \epsilon$. //

Lemma 6.4. *Let $(\mathbb{R}, \mathcal{M}, m)$ be the real numbers with the lebesgue measure. Let f be a measurable function on $[a,b]$ such that $|f| < \infty$ a.e., then $\forall \eta, \epsilon > 0$, there is a continuous function φ such that $m(\{x||f-\varphi| < \epsilon\}) < \eta$*

Proof. First given such an f we will show that f is bounded except on a set of small measure. Let $\epsilon > 0$. Let $E_n = \{x||f(x)| > n\}$. Then $[a,b] \supset E_1 \supset E_2 \supset \ldots$ and $m(\bigcap E_n) = 0$ which gives that there is an $M = n_0$ such that $|f(x)| < M$ off of a set E_{n_0} and $m(E_{n_0}) < \frac{\epsilon}{3}$. Now we will construct our continuous function in stages. Let n be such that $\gamma = \frac{2M}{n} < \epsilon$. Then for $k = 0, \ldots, n$ let $A_k = \{x| -M + (k-1)\gamma < f(x) \leq -M + k\gamma\}$. Then A_k are disjoint measurable sets and the function $\varphi_1 = \sum_{k=0}^{n} a_k \chi_{A_k}$ where $a_k = -M + k\gamma$, and $|\varphi_1 - f| < \epsilon$ where $|f(x)| < M$. Now φ_1 is definitly not continuous, however we can modify it to be. First for each k choose U_k open such that $m(U_k \triangle A_k) < \frac{\epsilon}{3n}$. Now revise $\varphi_2 = \sum_{k=0}^{n} a_k \chi_{U_k}$. Then $\{x|\varphi_2 \neq \varphi_1\} \subset \bigcup_{k=1}^{n} \{x \in [a,b]| \chi_{U_k} \neq \chi_{A_k}\} \subset \bigcup_{k=1}^{n} (U_k \triangle A_k)$ which gives that $m(\{x|\varphi_1 \neq \varphi_2\} \leq n\frac{\epsilon}{3n} = \frac{\epsilon}{3}$. Now φ_2 is a step function, thus there is a partition $a = x_0 < x_1 < \ldots < x_n = b$ such that φ_2 is constant c_k on each piece (x_{k-1}, x_k). Choose numbers $x_{k-1} < \alpha_k < \beta_k < x_k$ such that $\alpha_k - x_{k-1} < \frac{\epsilon}{bn}$ and $x_k - \beta_k < \frac{\epsilon}{bn}$. Now modify φ so that $\varphi(x_k) = 0$, and $\varphi(x) = c_k$ on $[\alpha_k, \beta_k]$. Extend $\varphi(x)$ linearly on $[x_{k-1}, \alpha_k]$ and $[\beta_k, x_k]$. Now φ is continuous and $\{x|\varphi \neq f\} \subset \bigcup([x_{k-1}, \alpha_k] \cup [\beta_k, x_k]$ which gives that $m(\{x|\varphi_2 \neq \varphi\}) \leq \sum_{k=1}^{n} \{(\alpha_k - x_{k-1}) + (x_k - \beta_k)\} < n * 2 * \frac{\epsilon}{6n} = \frac{\epsilon}{3}$. Summing these differences we see that $\{x|f \neq \varphi\} < \epsilon$. //

Theorem 6.5. *(Lusin) Let $(\mathbb{R}, \mathcal{M}, m)$ be the real numbers with the lebesgue measure. Let f be an measurable extended real valued function on $[a,b]$ such that $|f| < \infty$ a.e. Then given $\delta > 0, \exists \varphi$ continuous such that $m(\{x|f \neq \varphi\}) < \delta$.*

Proof. Let $\delta > 0$. Then by the previous lemma $\forall k \in \mathbb{N}$ let $A_k \in \mathcal{M}$ such that $A_k \subset [a,b]$ and $\varphi_k : [a,b] \to \mathbb{R}$ continuous such that $m(A_k) \leq 2^{-k}$ and $|f - \varphi_k| < \frac{1}{k}$ on $[a,b] \setminus A_k$. Then we have that $m(\bigcup_{n \geq k} A_n) \leq \sum_{n=k}^{\infty} 2^{-n} = 2^{-k+1}$. This gives that $lim_{k \to \infty} m(\bigcup_{n \geq k} A_n) = 0$. Thus we have that for $B = \bigcap_k \bigcup_{n \geq k} A_n$, then $m(B) = 0$. Further for $x \in [a,b] \setminus B$ we have there must be an element k such that $x \notin \bigcup_{n \geq k} A_n$ which gives that $x \notin A_n$ for $n \geq k$. Likewise for $n \geq k$ we have $|f(x) - \varphi_n(x)| < \frac{1}{n}$ by construction which gives that $\{\varphi_n\} \to f$ pointwise on $[a,b] \setminus B$. Now by Egoroff we can find a set C such that $m(C) < \delta$ for which the convergence is uniform on $[a,b] \setminus C$. Now we can find an open set U such that $C \subset U$ and $m(U) < \delta$. Let $F = [a,b] \setminus U$. Then F is closed, and $\{\varphi_n\}$ converge uniformally to φ' a continuous function on F. We can extend φ' to a continuous function φ on $[a,b]$ and $\{x|f \neq \varphi\} \subset C$ which gives $m(\{x|f \neq \varphi\}) < \delta$ //

7. Lebesgue Integral and Convergence Theorems

We are now ready to define the Lebesgue integral. To make some anaolgies we recall Riemann-integration and the following definition of it. One a closed bounded interval $[a,b]$ we can define a step function ψ to be a function such that there is a partition $a = x_0 < x_1 < \ldots < x_n = b$ such that $\psi(x) \cong c_k$ on (x_{k-1}, x_k). Then we can define $\int_a^b \psi dx = \sum_{k=1}^{n} c_k(x_k - x_{k-1})$. Given a function f on $[a,b]$ we can define $U \int_a^b f dx = inf \int_a^b \psi dx$ for $\psi(x) \geq f(x)$ and $L \int_a^b f dx = sup \int_a^b \psi dx$ for $\psi(x) \leq f(x)$. Then we have the equivalent defintion for Riemann integration of a function f provided $U \int_a^b f dx = L \int_a^b f dx$ and we define $\int_a^b f dx$ to be this common value. We can define the Lebesgue integral analgously however we must have a analogue to step functions.

Definition 7.1. *Given a measure space $(X, \mathcal{F}, \mu)$, a function $\varphi : D \to \mathbb{R}$ is simple if it is measurable and has finite range. Note that for a simple function φ with range $\{a_1, \dots, a_n\}$, we can give φ the canonical representation*

$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i} \text{ where } A_i = \{x | \varphi(x) = a_i\}$$

.

Such a defintion will not be useful unless these functions somehow approximate measurable functions.

Proposition 7.2. *Let $(X, \mathcal{F}, \mu)$ be a measure space and f a non-negative measurable function. Then there is a sequence $\{\varphi_n\}$ of simple functions that converge monotonically pointwise to f. Further if X is a $\sigma-$finite space, then these functions can be taken to be finitely supported.*

Proof. Let $E_{n,k} = \{x | k2^{-n} \leq f(x) < (k+1)2^{-n}\}$ and let $\varphi_n = 2^{-n} \sum_{k=0}^{2^{2n}} k \chi_{E_{n,k}}$. Then $\varphi_n \to f$ and $\varphi_n \leq \varphi_{n+1}$ as we only gain more sets as n increase. Now if X is $\sigma-$finite, let $X = \bigcup X_i$. Then each φ_n is only non-zero on a finitely collection of disjoint sets, say $E_1, \dots, E_m$ which must be contained in a finite collection $X_1, \dots, X_k$. Thus $\mu(X_1 \cup \dots \cup X_k) \leq \mu(X_1) + \dots + \mu(X_k) < \infty$. //

Notice that the geometry of this construction is to partition the y-axis to obtain the sets $E_k = \{x | k \leq f(x) < k+1\}$. This is the anaolgy between Riemann and Lebesgue integration. In the Riemann case we split the x-axis. In the Lebesgue case we split the y-axis. Notice that all the machinery so far has delt with giving sets that arise like E_k a notion of length (measure). In this spirit if we wanted to find an approximate for the integral of f we could just choose a test y-value and add up that value times the measure of E_k.

7.1. **Integration.** We can now define integration on simple functions and use the previous proposition to extend this definition to more general measurable functions.

Definition 7.3. *Given a complete space $(X, \mathcal{F}, \mu)$. For a given measurable set $E \in \mathcal{F}$ and simple function $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$, we define $\int_E \varphi d\mu = \sum_{i=1}^{n} a_i * \mu(A_i \cap E)$.*

Now we have an ambiguity of notation, so just as a convienence from here on out we will use $R \int_a^b f$ to denote Riemann interation. Likewise we have an ambiguity as φ could be represented in many different ways. However by finite addativity we can see that this defintion is invarient under change of representation for φ.

We have the following computational proposition:

Proposition 7.4. *Let $(X, \mathcal{F}, \mu)$ be a measure space and φ, ψ simple functions, then*

(1) $\int (a\varphi + b\psi) d\mu = a \int \varphi d\mu + b \int \psi d\mu$
(2) $\varphi \leq \psi \Rightarrow \int \varphi d\mu \leq \int \psi d\mu$.

Proof. (1) is true as sums are linear. The second is true as when $\varphi = \sum a_i \chi_{A_i}$ and $\psi = \sum b_i \chi_{B_i}$ then for any $x \in A_i$ we must have a set B_j such that $x \in B_j$ and $a_i \leq b_i$ which gives the result. //

We will extend our definition using the last proposition in the last section to non-negative measurable functions.

Definition 7.5. *Let $(X, \mathcal{F}, \mu)$ be a complete space and f a non-negative measureable function. Then we define $\int_E f d\mu = \sup_{\varphi \leq f}\{\int_E \varphi d\mu\}$ where φ is simple.*

A note that we could have defined integration of simple functions over non-complete spaces by $\int \varphi d\mu = \sum_{i=1}^n a_i \chi_{A_i}$ and $\int f d\mu = \sup_{\varphi \leq f}\{\int \varphi d\mu\}$ then define $\int_E \varphi d\mu = \int \varphi * \chi_E d\mu$. However restricting to complete measure spaces is not unreasonable. As we can always given a space $(X, \mathcal{F}, \mu)$ then we can construct a space $(X, \mathcal{F}_0, \mu_0)$ to be the same set where $\mathcal{F}_0 = \mathcal{F} \cup \{A \subset B | B \in \mathcal{F} \text{ and } \mu(B) = 0\}$ and defining μ_0 to be μ on $\mathcal{F}$ and 0 on any other set. Then this space is called the completion.

We should extend our definition a bit more as follows.

Definition 7.6. *Let $(X, \mathcal{F}, \mu)$ be a complete space and f measurable function. Then we can define $f^+(x) = max(f(x), 0)$ and $f^-(x) = max(-f(x), 0)$. Then each are non-negative measurable and we have $f = f^+ - f^-$ likewise $|f| = f^+ + f^-$. We can define $\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$*

Notice that now we have a notational ambiguity. For $\int f dx$ does this mean a Riemann integral or a lebesgue integral? We will henceforth denote $R\int_a^b f(x)dx$ for the Riemann integral of f over $[a, b]$.

We now present some basic properties of this integral.

Proposition 7.7. *Let $(X, \mathcal{F}, \mu)$ be a complete measure space. Then we have the following:*

1. *$0 \leq f \leq g$ for f, g measurable, then $\int_E f d\mu \leq \int_E g d\mu$*
2. *If $f \geq 0$ and $A \subset B$ both measurable then $\int_A f d\mu \leq \int_B f d\mu$ for measureable f.*
3. *$0 \leq f$ and $c < \infty$ then $\int_E cf d\mu = c \int_E f d\mu$*
4. *$f(x) = 0$ then $\int_E f d\mu = 0$ for any $E \in \mathcal{F}$*
5. *$\mu(E) = 0$ then $\int_E f d\mu = 0$ for any f measurable*
6. *If $a \leq f(x) \leq b$ and $\mu(E) < \infty$ then $a\mu(E) \leq \int_E f d\mu \leq b\mu(E)$.*

Proof. (1) We have that any simple function $\varphi \leq f$ gives $\varphi \leq g$.

(2) If $A \subset B$ then for any simple function $0 \leq \varphi \leq f$ then we have $\int_A \varphi d\mu \leq \int_B \varphi d\mu$ by defintion.

(3) Let $c < \infty$, and φ a simple function $\varphi \leq f$. Then $\int_E c\varphi d\mu = c \int_E \varphi d\mu$.

(4) Let $E \in \mathcal{F}$ and $f(x) = 0$. Then any non-negative simple function φ must be 0.

(5) Let f be measurable. Then for any simple function φ we have $\int_E \varphi d\mu = 0$ as $\mu(E) = 0$.

(6) Let $a \leq f(x) \leq b$ and $\mu(E) < \infty$. Then the result holds for any simple function $a \leq \varphi \leq b$ which we can choose by approximation.

//

We must postpone linearity until later as it will require some of the convergence theorems.

7.2. **Bounded measurable functions on $(\mathbb{R}, \mathcal{M}, m)$.** While we don't need this section to get to the meat of the integration theory, namely the next section. The methods here will mostly generalize for complete $\sigma-$finite spaces.

We start with the following propositon:

Proposition 7.8. *Let $E \in \mathcal{M}$ be such that $m(E) < \infty$. Then a bounded function f is measurable iff $\inf_{f\leq\psi}\int_E \psi(x)dx = \sup_{\varphi\leq f}\int_E \varphi(x)dx$ for simple functions φ and ψ.*

Proof. Let f be a bounded function that is measurable, $|f| \leq M$. Then for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ let $E_{n,k} = \{x | \frac{(k-1)M}{n} \leq f(x) \leq \frac{kM}{n}\}$ be a partition of the y-axis. Then each $E_{n,k}$ is measurable and $\sum_{-n}^{n} m(E_{n,k}) = m(E)$. Now we define $\psi_n = \frac{M}{n}\sum_{k=-n}^{n} k\chi_{E_{n,k}}$ and $\varphi_n = \frac{M}{n}\sum_{k=-n}^{n}(k-1)\chi_{E_{n,k}}$. Then we have that $\varphi_n \leq f \leq \psi_n$ on E and $\psi_n - \varphi_n = \frac{M}{n}\chi_E$. This gives the result. Conversly let f be bounded such that $inf_{f\leq\psi}\int_E \psi(x)dx = \sup_{\varphi\leq f}\int_E \varphi(x)dx$. Then let $\varphi^* = sup_{\varphi\leq f}\varphi_n$ and $\psi^* = inf_{\psi\geq f}\psi_n$ then φ^* and ψ^* are both measurable and we have $\varphi^* \leq f(x) \leq \psi^*$. Let $\triangle = \{x|\varphi^* < \psi^*\}$, and $\triangle_k = \{x|\varphi^* < \psi^* - \frac{1}{k}\}$. Then $m(\triangle_k) \leq \frac{k}{n}$. By letting n increase without bound, we have $m(\triangle_k) = 0$ for each k thus $m(\triangle) = 0$, thus $f = \varphi^*$ a.e. and so f is measurable. //

This common value is of course the lebesgue integral of f. Likewise not the similarity of this proposition and the definition of Riemann integrals using step functions. As a matter of fact, All step functions are just specific simple functions. This restriction is why the lebesgue integral is more general than the Riemann integral.

Proposition 7.9. *Let f be a bounded function on $[a, b]$. Then f if is Riemann integrable then it is measurable and $R\int_a^b f dx = \int_a^b f dx$.*

Proof. This is clear as simple functions are step functions, so $RL\int_a^b f dx \leq \sup_{\varphi\leq f}\int_a^b \varphi dx \leq \inf_{\psi\geq f}\psi dx \leq RU\int_a^b f dx$. //

We can easily show linearity for the bounded measurable functions as follows:

Proposition 7.10. *Let f, g be bounded measurable functions and $E \in \mathcal{M}$ be such that $m(E) < \infty$. Then $\int_E (f+g)dx = \int_E f dx + \int_E g dx$.*

Proof. Let $f \leq \psi_1$, $g \leq \psi_2$ where ψ_1 and ψ_2 are simple functions. Then we have that $\int_E (f+g)dx \leq \int_E(\psi_1+\psi_2)dx = \int_E \psi_1 dx + \int_E \psi_2 dx$. Now we can take the infimum of the right side to get $\int_E(f+g)dx \leq \int_E f dx + \int_E g dx$. Now we can make the dual argument with φ_1 and φ_2 simple functions such that $\varphi_1 \leq f$ and $\varphi_2 \leq g$. //

Now we have a 'convergence' theorem for bounded measurable functions. Theorems of this type are the meat of the integration theory as the results of such tend to give the closure of integrals under pointwise convergence theorems.

Theorem 7.11. *(Bounded Convergence Theorem) Let $\{f_n\}$ be a sequence of measureable functions such that $|f_n(x)| \leq M$ for each n and x. Then if $\{f_n\} \to f$, then $\int_E f_n dx \to \int_E f dx$.*

Proof. Given $\epsilon > 0$ we need to choose N such that for each $n \geq N$ we have $\int_E (f_n - f)dx < \epsilon$. First we will use Egoroff to obtain uniform convergence except on a small set. Egoroff gives that we can choose δ such that for any set $A \subset E$ such that $m(A) < \delta$ then we can have uniform convergence on $E \setminus A$. Given this set A we can find N such that $|f_n - f| < \frac{\epsilon}{2m(E)}$ on $E \setminus A$. This gives that $\int_{E\setminus A} |(f_n - f)|dx < \frac{\epsilon}{2m(E)} m(E) = \frac{\epsilon}{2}$. Now which every δ we choose, we will have

$m(A) < \delta$ and so $\int_A |f_n - f|dx \leq \int_A 2Mdx \leq 2M\delta$. So the nice choice for $\delta < \frac{\epsilon}{4M}$. Then we would have $\int_A |f_n - f|dx \leq 2M\delta \leq \frac{\epsilon}{2}$. Putting this all together for $n \geq N$ we have $|\int_E (f_n - f)dx| \leq \int_E |f_n - f|dx \leq \int_A |f_n - f|dx + \int_{E\setminus A} |f_n - f|dx \leq 2M\delta + \frac{\epsilon}{2m(E)} * m(E) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. //

A nice result to mention here is the following theorem.

Theorem 7.12. *(Lebesgue Theorem) A bounded function f on $[a, b]$ is Riemann integrable iff f is continuous almost everywhere*

The proof of this theorem is in the exercise section below.

7.3. **Convergence Theorems.** We now prove the big theorems in the integration theory. We begin with the following:

Theorem 7.13. *(Monotone Convergence) Let $(X, \mathcal{F}, \mu)$ be a space. Let $\{f_n\}$ be a montone sequence of non-negative measurable functions and $\{f_n\} \to f\mu-a.e.$ Then $\int f d\mu = lim \int f_n d\mu$.*

Proof. Let $(X, \mathcal{F}, \mu)$ be a space. Since $f_n \leq f_{n+1}$ we have that $\int f_n d\mu \leq \int f_{n+1} d\mu$, we have that $lim \int f_n d\mu \to \alpha \in [0, \infty]$. Now since $\int f_n d\mu \leq \int f d\mu$ we have that $\alpha \leq \int f d\mu$. Now let φ be a simple function such that $\varphi \leq f$. Let $c \in (0, 1)$. Let $A_n = \{x | f_n(x) \geq c\varphi(x)\}$. Since $f_n(x)$ increases to $f(x)$ and $c < 1$ we have $A_1 \subset A_2 \subset \ldots$, and $\bigcup A_n = X$. Then we have $c \int_{A_n} \varphi d\mu \leq \int_{A_n} f_n d\mu \leq \int f_n d\mu$. Taking limits with respect to n, we have that $\alpha \geq c \int \varphi d\mu$. Since $c \in (0, 1)$ was arbitrary, we have that $\int \varphi d\mu \leq \alpha$. And so taking suprema we find that $\int f d\mu \leq \alpha$. Which gives the result. //

We should note we can use the results above for bounded measurable functions to give a different proof in the case of a $\sigma-$finite complete measure space.

Proof. Let $(X, \mathcal{F}, \mu)$ be a $\sigma-$finite complete measure space. Let f be a non-negative measurable function. Then we can approximate f with a sequence $\{h_n\}$ of bounded finitely supported simple functions such that $h_n \leq f_n$. This gives that $\int h_n d\mu \leq \int f_n d\mu$ and taking supremums and the Bounded Convergence theorem we have $\int f d\mu \leq \sup \int f_n d\mu$. Likewise since $f_n \leq f$ we have that $\int f_n d\mu \leq \int f d\mu$ which gives that $\sup \int f_n d\mu \leq \int f d\mu$, thus $\int f d\mu = \sup \int f_n d\mu = lim \int f_n d\mu$. //

Lemma 7.14. *(Fatou) Let $(X, \mathcal{F}, \mu)$ be a space. Then for $\{f_n\}$ a sequence of non-negative measurable functions we have $\int \overline{\lim} f_n \leq \overline{\lim} \int f_n$.*

Proof. Let $(X, \mathcal{F}, \mu)$ be a space, let $f = lim f_n = \sup_{n\geq 1} inf_{n\geq k} f_k$. Let $g_n = inf_{k\geq n} f_k$. Then g_n is an monotone sequence going to $\underline{\lim} f_n$. So by monotone convergence theorem we have $lim \int g_n d\mu = \int \underline{\lim} f_n d\mu$. Now $g_n \leq f_k$ for $k \geq n$, thus $\int g_n d\mu \leq \int f_k d\mu$. Taking infema we have $\int g_n d\mu \leq inf_{k\geq n} \int f_k d\mu$. Now taking suprema we have $\int \underline{\lim} f_n d\mu = \sup_n \int g_n d\mu \leq \sup_n inf_{k\geq n} \int f_k d\mu = \underline{\lim} \int f_k d\mu$. //

We should note that these two results are equivalent to one another.

Theorem 7.15. *(Monotone Convergence) Let $(X, \mathcal{F}, \mu)$ be a space. Let $\{f_n\}$ be a montone sequence of non-negative measurable functions and $\{f_n\} \to f\mu-a.e.$ Then $\int f d\mu = lim \int f_n d\mu$.*

Proof. Let $(X, \mathcal{F}, \mu)$ be a space. Let $f_n \nearrow f$ be a sequence of non-negative measurable functions. Then we have by Fatou that $\int f \leq \underline{\lim} f_n$ and since $f_n \leq f$ we have that $\int f_n \leq \int f$ so $\underline{\lim} \int f_n \leq \int f$, so we have $\int f_n = \int f$. //

Now we can obtain the result:

Proposition 7.16. *Given $(X, \mathcal{F}, \mu)$, let f, g be non-negative measurable functions. Then $\int_E (f+g) d\mu = \int_E f d\mu + \int_E g d\mu$.*

Proof. Let $(X, \mathcal{F}, \mu)$ be a space, let φ_n be a simple approximation for f and ψ_n be a simple approximation for g. Then we have $\varphi_n + \psi_n$ then we have that $\varphi_n + \psi_n \to f + g$ thus we have $\int f d\mu + \int g d\mu = \int \varphi_n d\mu + \int \psi_n d\mu = \int (\varphi_n + \psi_n) d\mu = \int (f+g) d\mu$ by the Monotone convergence theorem. //

We have the following quick corollary

Corollary 7.17. *Given $(X, \mathcal{F}, \mu)$ Then for disjoint measurable sets A, B and measurable function f, we have that $\int_{A \cup B} f = \int_A f + \int_B f$*

Proof. We have that $\chi_{A \cup B} = \chi_A + \chi_B$ and so $\int_{A \cup B} f = \int f \chi_{A \cup B} = \int f(\chi_A + \chi_B) == int f \chi_A + \int f \chi_B = \int_A f + \int_B f$ //

We now make the following defintion.

Definition 7.18. *Given $(X, \mathcal{F}, \mu)$, let f be a measureable function. Then f is Lebesgue integrable on $E \in \mathcal{F}$ provided $\int_E f d\mu < \infty$.*

Integrable functions have many nice properties. For example:

Corollary 7.19. *Given $(X, \mathcal{F}, \mu)$, let f be a non-negative integrable function on a set $E \in \mathcal{F}$. Then for each $\epsilon > 0$ there is a $\delta > 0$ such that for any set $\mu(A) < \delta$ then $\int_A f < \epsilon$.*

Proof. Let $(X, \mathcal{F}, \mu)$ be a space, f non-negative integrable, and let $\epsilon > 0$. Let $f_n = min f, n$, then f_n is a monotonic sequence going to f. By montone convergence we have then that $lim \int_E f_n d\mu = \int_E f d\mu$. Therefore we can find N such that $\int_E f_n d\mu > \int_E f d\mu - \frac{\epsilon}{2}$. Since f is integrable we have then that $\int_E (f_n - f) d\mu < \frac{\epsilon}{2}$. Then for any set $A \subset E$ such that $\mu(A) < \delta$ we have $\int_A f d\mu = \int_A (f - f_n) d\mu + \int_A f_n < \frac{\epsilon}{2} + N\mu(A)$ so if we need to choose $\delta < \frac{\epsilon}{2N}$, then we have $\int_A f d\mu = \int_A (f - f_n) d\mu + \int_A f_n < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ //

Corollary 7.20. *Let $(X, \mathcal{F}, \mu)$ be a space, let $\{f_n\}$ be a sequence of non-negative measurable functions and $f = \sum_{n=1}^{\infty} f_n$. Then we have $\int f dx = \sum_{n=1}^{\infty} \int f_n dx$.*

Proof. Let $g_k = \sum_{n=1}^{k} f_n$. Then g_k increases monotonically to $\sum_{n=1}^{\infty} f_n$. Therefore $\sum_{n=1}^{\infty} \int f_n d\mu = lim \int g_k d\mu = \int \sum_{n=1}^{\infty} f_n d\mu$. //

Using integrable functions we can get a stronger convergence theorem.

Theorem 7.21. *(Lebesgue Dominated Convergence Theorem) Let $(X, \mathcal{F}, \mu)$ be a space,let $\{f_n\}$ be a sequence of measureable functions such that $|f_n| \leq g$ and g is integrable. Then if $\{f_n\} \to f \mu - a.e.$ Then we have $\int f = lim \int f_n$.*

Proof. Let $(X, \mathcal{F}, \mu)$ be a space. Since $|f_n| \leq g$ we have that $(g - f_n) \geq 0$. Then we have a sequence of non-negative functions, we have by Fatou that $\int g d\mu - \int f d\mu = \int (g - f) d\mu \leq \overline{\lim} \int (g - f_n) d\mu = \overline{\lim} (\int g d\mu - \int f_n d\mu) \leq \overline{\lim} \int g d\mu + \underline{\lim} \int (-f_n) d\mu) =$

$\int f d\mu - \overline{\lim} \int f_n d\mu$. This gives that $\int g d\mu - \int f d\mu \leq \int g d\mu - \overline{\lim} \int f_n d\mu$ since g is integrable we have then that $\overline{\lim} \int f_n d\mu \leq \int f d\mu$. Now we have also that $g + f_n \geq 0$ which gives that this also is a sequence of non-negative functions. Therefore again by Fatou we have that $\int g d\mu + \int f d\mu = \int (g+f) d\mu \leq \overline{\lim} \int (g+f_n) d\mu = \overline{\lim}(\int g d\mu + \int f_n d\mu) \leq \overline{\lim} \int g d\mu + \overline{\lim} \int f_n d\mu = \int g d\mu + \overline{\lim} \int f_n d\mu$. Then again since g is integrable we have that $\int f d\mu \leq \overline{\lim} \int f_n d\mu$. This gives the result.

//

The same proof using g replaced by the right g_n essentially give the following generalization:

Theorem 7.22. *(Lebesgue Dominated Convergence Theorem) Let $(X, \mathcal{F}, \mu)$ be a space,let $\{f_n\}$ be a sequence of measureable functions such that $|f_n| \leq g_n$ and g_n is integrable. If $\{f_n\} \to f \mu-$a.e. and $\{g_n\} \to g\mu-$a.e. Then we have $\int g dx = lim \int g_n dx \Rightarrow \int f dx = lim \int f_n dx$.*

The following example shows a sequence which does not meet the hypothesis for Lebesgue Dominated Convergence Theorem.

Example 7.23. *Consider $(\mathbb{R}, \mathcal{M}, m)$. Let $f_n = n\chi_{(0,\frac{1}{n})}$. Then $f_n \geq 0$ and $f_n \to 0$. However $\int f_n dx = 1$. However notice that no integrable g that bounds all the f.*

Notice also in the previous example that the sequence is decreasing, that is $f_n \searrow 0$. This gives that we cannot have a decreasing version of the monotone convergence theorem without some hypothesis of integrability

8. General Mesaure

Up to this point we have only been dealing with one fixed measure on a fixed $\sigma-$algebra. Now we will consider what kinds of relationships can exists between different measures on the same $\sigma-$algebra, specifically when they have the same measureable sets. For notational purposes we will denote by a $(X, \mathcal{F})$ a set X and a $\sigma-$algebra $\mathcal{F}$ such that $\mathcal{F} \subset X$.

We are concenred with the following two relationships.

Definition 8.1. *Given two measures λ and μ on a given $\sigma-$algebra $\mathcal{F}$ on a set X. Then we say that λ is absolutely continuous with respect to μ (written $\lambda << \mu$) provided that for any $E \in \mathcal{F}$ such that $\mu(E) = 0$ then $\lambda(E) = 0$. The two measures are mutually singular provided there exits $A, B \in \mathcal{F}$ disjoint such that $X = A \cup B$ and $\mu(A) = \lambda(B) = 0$. This is written $\mu \perp \lambda$.*

In this definition, absolutely continuous essentially says that the one measure λ repsects the sets that don't matter with respect to μ. One can think of this that λ does not try to give validity to any set which μ has already disregarded, in this sense the two are compatible. The other relationship says that neither measure is positive at the same time, hence both measures leave each other alone.

We will show some examples of these relationships.

Example 8.2. *Given a measure space $(X, \mathcal{F}, \mu)$ and an integrable function f, then we can form a new measure $\lambda(E) = \int_E f d\mu$ on $\mathcal{F}$. We have that $\lambda << \mu$.*

Example 8.3. *Consider $(\mathbb{R}, \mathcal{M})$ where $\mathcal{M}$ are the lebesgue measureable subsets of $\mathcal{R}$. Then define $\mu(E \cap (\infty, 0) = m(E \cap (\infty, 0))$ where m is the regular lebesgue measure, and $\lambda(E \cap [0, \infty))$ be the counting measure. Then we have that $\mu \perp \lambda$.*

The reason we call $\lambda << \mu$ absolutely continuous is that from above we have for integrable f then $\lambda(E) = \int_E f d\mu < \epsilon$ for $\mu(E) < \delta$.

To fully describe the relationships that can occur in general we will need to extend our definition of measure slightly and develop some techniques for dealing with situations when our measures aren't always positive.

8.1. **Signed Measures, Hahn and Jordan Decompositions.** Consider the situtation when we have two measures μ, λ on the same $\sigma-$algebra $\mathcal{F}$. Then we would have naturally that $\mu + \lambda$ is a measure, however what happens when we consider $\mu - \lambda$. Well given the right set this quantity could be negative. Further we could have some arithemtical problems if either measure measures sets to be infinite. We can extend our definition of measure slightly to allow for measures to take on negative values.

Definition 8.4. *Consider $(X, \mathcal{F})$. Then we call a set function $\mu : \mathcal{F} \to [-\infty, \infty]$ a (signed) measure provided*

(1) *μ omits either ∞ or $-\infty$.*
(2) *$\mu(\emptyset) = 0$.*
(3) *$\mu(\bigcup E_i) = \sum \mu(E_i)$ for $\{E_i\} \subset \mathcal{F}$ disjoint and the series $\sum \mu(E_i)$ converges absolutely.*

The triple $(X, \mathcal{F}, \mu)$ where μ is a signed measure is called a signed measure space.

Now for a given signed measure space, we can reference sets in the $\sigma-$algebra by how they are measured.

Definition 8.5. *Let $(X, \mathcal{F}, \mu)$ be a signed measure space. Then a set $A \in \mathcal{F}$ is called positive provided $\forall E \subset A$ such that $E \in \mathcal{F}$ then $\mu(E) \geq 0$. Dually a set is negative if $\mu(E) \leq 0$, and a set is a null set if E is both positive and negative.*

We have that measureable subsets of a positive, negative, or null sets are respecively so, and likewise so are countable unions. We pause to give some examples of signed measure spaces.

Example 8.6. *Consider $X = \mathbb{N}$ and $\mathcal{F} = \wp(\mathbb{N})$. Then define $\mu(n) = 2^{-n} - \frac{1+(-1)^n}{2}$, and extend by taking sums. Consider $A = \{$ odd numbers $\}$ Then $\mu(A) = \sum 2^{-(2n+1)} = \frac{2}{3}$, and for $B = \{$ even numbers $\}$, then $\mu(B) = -\infty$.*

Naturally many examples can come about by simple subtracting two measures μ and λ to obtain a signed measure $\nu = \mu - \lambda$ provided one of the measures is finite. The next couple theorems basically say that this is the only way to obtain signed measures.

Theorem 8.7. *(Hahn) Given a signed measure space $(X, \mathcal{F}, \mu)$ then there is a positive set A and a disjoint negative set B such that $X = A \cup B$.*

Proof. Without loss of generality let $\mu(A) \neq -\infty$ for any $A \in \mathcal{F}$. Let $\alpha = \sup_{A \in \mathcal{F} \text{positive}} \mu(A)$. Then we have that $0 \leq \alpha$. Then we can find $\{A_i\}$ a sequence of sets such that $\alpha = \lim \mu(A_i)$. Let $A = \bigcup A_i$. Then A is positive and $\mu(A) \leq \alpha$. We have also that $A \setminus A_i \subset A$ so $\mu(A \setminus A_i) \geq 0$. This gives that $\mu(A_i) \leq \mu(A_i) + \mu(A \setminus A_i) = \mu(A)$ this gives that $\mu(A) \geq \alpha$ and so $\mu(A) = \alpha$. Let $B = X \setminus A$. Then if $E \subset B$ is positive then $\alpha + \mu(E) = \mu(A) + \mu(E) = \mu(E \cup A) \leq \mu(A) = \alpha$, so $\mu(E) = 0$, and we have that B must be negative. //

The pair (A, B) given by the theorem is called the Hahn decomposition, and unfortunately this is not unique. The set E in the proof of the theorem could be null and so lie in either A or B without affecting the decomposition. We give an example to illustrate this.

Example 8.8. *Let $X = \{a, b, c, d, e\}$ and $\mathcal{F} = \wp(X)$. Then let $f : X \to \mathbb{R}$ be $f(a) = f(b) = -1$ and $f(d) = f(e) = 1$ and $f(c) = 0$. Let $\mu(E) = \sum_{x \in E} f(x)$. Then μ is a signed measure, and we have that both $X_1^+ = \{c, d, e\}, X_1^- = \{a, b\}$ and $X_2^+ = \{d, e\}, X_2^- = \{a, b, c\}$ are both valid Hahn decompositions.*

Given a Hahn decomposition (X^+, X^-) of a signed measure space $(X, \mathcal{F}, \mu)$ consider $\mu^+(E) = \mu(E \cap X^+)$ and $\mu^-(E) = -\mu(E \cap X^-)$ then we have that both μ^+ and μ^- are measures and $\mu(E) = \mu^+(E) - \mu^-(E)$. These two measures are in fact mutually singular ($\mu^+ \perp \mu^-$) by definition. We will summarize this in the following definition.

Definition 8.9. *Given a signed measure space $(X, \mathcal{F}, \mu)$ then we have that the pair μ^+ and μ^- is the unique pair of mutually singular measures such that $\mu = \mu^+ - \mu^-$ is called the Jordan decomposition.*

Now given a signed measure and a Jordan decomposition μ^+ and μ^- then we have that $\mu = \mu^+ - \mu^-$ however we can construct a related positive measure as follows.

Definition 8.10. *Given a signed measure space $(X, \mathcal{F}, \mu)$ then we define the total variation measure $|\mu|$ given the Jordan decomposition μ^+ and μ^- as $|\mu| = \mu^+ + \mu^-$. Further we define the norm of the measure $||\mu|| = |\mu|(X)$.*

In the above example let X_1^+ and X_1^- be the first Hahn decomposition, then $\mu^+(E) = |E \cap \{d, e\}|$ and $\mu^-(E) = |E \cap \{a, b\}|$. Then we have that $|\mu| = |E \cap \{a, b, d, e, \}|$ and $||\mu|| = 4$.

We may not have uniqueness of Hahn decompositions, however we do have that measures which 'share' Hahn decompositions in some fasion must be the same.

Lemma 8.11. *Let μ, ν, λ be measures on a pair $(X, \mathcal{F})$ such that $\mu(X) < \infty$ and $\nu << \mu$ and $\lambda << \mu$. Then suppose that $\forall \alpha \in \mathbb{R}$ we have $\nu - \alpha\mu$ and $\lambda - \alpha\mu$ have the same Hahn decompositions. then $\nu = \lambda$.*

Proof. $\forall \alpha \in \mathbb{R}$ let (A_α, B_α) be a common Hahn decomposition for $\nu - \alpha\mu$ and $\lambda - \alpha\mu$. Fix N. Then for k let $E_{N,k} = B_{\frac{k}{N}} \setminus B_{\frac{k-1}{N}}$. If $C \in \mathcal{F}$, and $C \subset E_{N,k}$ then $(\nu - \frac{k}{N}\mu)(C) \leq 0 \leq \nu - \frac{k-1}{N}\mu)(C)$ so $\frac{k-1}{N}\mu(C) \leq \nu(C) \leq \frac{k}{N}\mu(C)$. Likewise we have $\frac{k-1}{N}\mu(C) \leq \lambda(C) \leq \frac{k}{N}\mu(C)$. Let $E_{N,\infty} = X \setminus \bigcup E_{N,k}$, and let $C \in \mathcal{F}$. Then we can write $C = \bigcup(C \cap E_{N,k}) \cup (C \cap E_{N,\infty})$. We have then that $\nu(E_{N,k}) - \frac{1}{N}\mu(E_{N,k}) \leq \lambda(E_{N,k}) \leq \nu(E_{N,k}) + \frac{1}{N}\mu(E_{N,k})$. If $\mu(E_{N,\infty}) = 0$ then we have that $\nu(E_{N,\infty}) = 0 = \lambda(E_{N,\infty})$. If $\mu(E_{N,\infty})$ then since $E_{N,\infty}$ is positive for $(\nu - \alpha\mu)$ and $(\lambda - \alpha nu)$ for each α we have that $\nu(E_{N,\infty}) = \lambda(E_{N,\infty})$. Hence we have that $\nu(C) - \frac{1}{N}\mu(C) \leq \lambda(C) \leq \nu(C) + \frac{1}{N}\mu(C)$. Since $\mu(C) < \infty$ and N was arbitrary we have that $\nu(C) = \lambda(C)$ and the lemma is proved. //

8.2. **Complex Measures.** In this last section we saw some of the extension of measures into ranges other than $\mathbb{R}^+$ namely $\mathbb{R}$. One could beg the question of what does it mean to have a measure with a range of $\mathbb{C}$. We can extend just as we did in the case of signed measurs, only now we do not have to worry about ∞. We make the following definition

Definition 8.12. *Given $(X, \mathcal{F})$, then a set function $\mu : \mathcal{F} \to \mathbb{C}$ is a complex measure provided*

(1) $\mu(\emptyset) = 0$.
(2) $\mu(\bigcup E_i) = \sum \mu(E_i)$ *for* $\{E_i\} \subset \mathcal{F}$ *disjoint and the series* $\sum \mu(E_i)$ *converges absolutely.*

We well call then a triple $(X, \mathcal{F}, \mu)$ a complex measure space provided μ is a complex measure.

Recall now the decompositions for a signed measure space $(X, \mathcal{F}, \mu)$ and a Hahn decomposition X^+ and X^-. Then we have that the total variation measure $|\mu|$ is a positive measure such that for any measurable E, then $|\mu(E)| = |\mu^+(E) - \mu^-(E)| \leq \mu{+}(E) + \mu^-(E) = |\mu|(E)$ and further this is the smallest such in the following sense. Given a positive measure λ that satisfies the same conditions, then we have that for any set E that $E = E \cap X^+ \cup E \cap X^-$. So $|\mu|(E \cap X^+) = \mu^+(E \cap X^+) \leq \lambda(E \cap X^+)$ and $|\mu|(E \cap X^-) = \mu^-(E \cap X^-) = |\mu(E \cap X^-)| \leq \lambda(E \cap X^-)$ so we would have that $|\mu|(E) \leq \lambda(E)$. So we note that $|\mu|(E) = |\mu(E \cap X^+)| + |\mu(E \cap X^-)|$. We also note that $E \cap X^+$ and $E \cap X^-$ are the maximal way to split E in the sense that if we write $E = \cup E_i$ then $|\mu|(E) \geq \sum |\mu(E_i)|$. Inspired by this we can similarly then given a complex measure space $(X, \mathcal{F}, \mu)$, the total variation measure.

Definition 8.13. *Given a complex measure space $(X, \mathcal{F}, \mu)$ then the total variation measure of μ is given by $|\mu|(E) = \sup \sum |\mu(E_i)|$ where the supremum is taken over all $\bigcup E_i = E$.*

We have then that $|\mu|$ again satisfies the two conditions afore mentioned. We have then also that we can define a Jordan decomposition $\mu^+ = \frac{1}{2}(|\mu| + \mu)$ and $\mu^- = \frac{1}{2}(|\mu| - \mu)$ as $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$. This defintion is consistent with the Jordan composition when μ is a finite signed measure.

We can now extend our definition of absolute continuity to complex (and hence signed) measures by saying that for complex measures, λ, μ then we say $\lambda << \mu$ when $|\lambda| << |\mu|$. We note that for any complex measure μ then $\mu << |\mu|$, that is to say if $|\mu|(E) = 0$ then $\mu(E) = 0$.

8.3. **Radon-Nikodym.** The main result of this section in many ways can be called the fundemental theorem of measure theory. We will also be able to classify the relationship between two measures on the same $\sigma-$algebra.

In the example given above of we obtained an absolutely continuous measure λ given a measure μ and a integralbe function f by $\lambda(E) = \int_E f d\mu$. The next theorem gives that for $\sigma-$finite spaces, this is the only way that this can happen.

Theorem 8.14. *(Radon-Nikodym) Let $(X, \mathcal{F}, \mu)$ be a $\sigma-$finite positive space and and let λ be a [3] measure on $\mathcal{F}$ such that $\lambda << \mu$. Then there is a unique almost everywhere non-negative function f such that*

$$\lambda(E) = \int_e f d\mu$$

.

[3]The case of where λ is a complex measure is handled in eloquantly [4], however this proof went beyond the scope of the course offered this semester, in this case we lose the non-negativite condition and gain integrability

Proof. Extending from the finite case to the $\sigma-$finite case is easy by taking and $X = \bigcup E_n$ where $\mu(E_n) < \infty$ and $f = \prod f_n \chi_{E_n}$ with f_n given by the finite case. So without loss of generality let $\mu(X) < \infty$. Now consider $\lambda - \alpha\mu$ is a signed measure for each rational α. Let (A_α, B_α) be a Hahn decomposition and take $A_0 = X$ and $B_0 = \emptyset$. Now $B_\alpha \setminus B_\beta = B_\beta \cap A_\beta$, so we have that $(\lambda - \alpha\mu)(B_\alpha \setminus B_\beta) \leq 0$ and $(\lambda - \beta\mu)(B_\beta \setminus B_\alpha) \geq 0$, so if $\beta > \alpha$ then $\mu(B_\alpha \setminus B_\beta) = 0$. Thus we can find a $\mu-$measurable function f such that $f \geq \alpha \quad \mu-$a.e. on A_α and $f \leq \alpha \quad \mu-$ on B_α. Since $B_0 = \emptyset$ we may take f to be non-negative. Let $\nu(E) = \int_E f d\mu$. Now we must show that $\lambda = \nu$. This was solved for us by the previous lemma. //

Here is quick example to show that $\sigma-$finiteness is important.

Example 8.15. *Consider $X = \{a\}$ and $\mathcal{F} = \wp(X)$. Define $\mu(\emptyset) = 0$ nad $\mu(X) = \infty$, also define $\lambda(\emptyset) = 0$ and $\lambda(\emptyset) = 1$. Then we have that $\lambda << \mu$ but $\lambda(E) \neq \int_E f d\mu$ for any f.*

We call f the Radon-Nikodym derivative and it is denoted $f = [\frac{d\lambda}{d\mu}]$. Using this theorem we will prove the following.

Theorem 8.16. *(Lebesgue decomposition) Let $(X, \mathcal{F}, \mu)$ be a $\sigma-$finite measure space, and λ another $\sigma-$finite measure on $\mathcal{F}$. Then we can write $\lambda = \lambda_a + \lambda_s$ uniquely such that $\lambda_s \perp \mu$ and $\lambda_a << \mu$.*

Proof. Since μ and λ are both $\sigma-$finite so is $\nu = \mu + \lambda$, and since μ and λ are absolutely continuous with respect to ν we can find a Radon-Nikodym derivatives f, g such that $\mu(E) = \int_E f d\nu$ and $\lambda(E) = \int_E f d\nu$. Let $A = \{x | f(x) > 0\}$ and $B = \{x | f(x) = 0\}$. Then $X = A \cup B$ disjointly and if $\mu(B) = 0$ then we can define $\lambda_s(E) = \lambda(E \cap B)$. We have that $\lambda_s(A) = 0$ and so $\lambda_s \perp \mu$. Further Let $\lambda_a(E) = \lambda(E \cap A) = \int_{E \cap A} g d\nu$. Then we have htat $\lambda = \lambda_a + \lambda_s$. Given E such that $\mu(E) = 0$ then we have that $\mu(E) = 0 = \int_E f d\nu$ or that f must be 0 $\nu-$a.e. on E. Since $f > 0$ on $A \cap E$ we have that $\nu(A \cap E) = 0$ which gives that $\lambda(A \cap E) = 0$ and finally $\lambda_a(A \cap E) = 0$. //

9. Fundamental Theorem

We will now discuss some generalities of the fundemental theorem of Calculus. Recall that for a differentiable function f then $\int_a^b f'(x)dx = f(b) - f(a)$ for Riemann integration and likewise $\frac{d}{dx} \int_a^x f(y)dy = f(x)$ also for Riemann integrable functions. We ask the question of when does this notion extend for the lebesgue integral on $(\mathbb{R}, \mathcal{M}, m)$.

We first start with some definitions:

Definition 9.1. *A collection of intervals $\{I_n\}$ is said to be a Vitali covering of a set E (or a covering of E in the sense of Vitali), provided $E \subset \bigcup I_n$ and for each $\epsilon > 0$ and $x \in E$ then we can find $I \in \{I_n\}$ such that $x \in I$ and $l(I) < \epsilon$.*

Lemma 9.2. [4] *Let E be a set such that $m^*(E) < \infty$, and $\{I_n\}$ be a Vitali covering. Then given $\epsilon > 0$ there is a collection $\{I_1, \ldots, I_N\} \subset \{I_n\}$ such that $m^*(E \setminus \bigcup_{j=1}^N I_j) < \epsilon$.*

[4] The proof of this lemma is difficult and technical, so it is omited

9.1. **Functions Bounded Variation.** We now consider certain but important class of functions.

Definition 9.3. *Let f be a function defined on $[a, b]$. Then we define:*

(1) $P_a^b(f) = sup_{P \text{ partition } [a,b]} \sum_{i=1}^k [f(x_i) - f(x_{i-1})]^+$
(2) $N_a^b(f) = sup_{P \text{ partition } [a,b]} \sum_{i=1}^k [f(x_i) - f(x_{i-1})]^-$
(3) $T_a^b(f) = sup_{P \text{ partition } [a,b]} \sum_{i=1}^k |f(x_i) - f(x_{i-1})|$

We call $P_a^b(f)$ the positive variation, $N_a^b(f)$ the negative variation and $T_a^b(f)$ the total variation. We have the following:

Definition 9.4. *Let f be a function defined on $[a, b]$ then we say that f is of bounded variation provided $T_a^b(f) < \infty$.*

Consequences of bounded variation are far reaching:

Lemma 9.5. *Let f be a function of bounded variation. Then $T_a^b(f) = P_a^b(f) + N_a^b(f)$ and $f(b) - f(a) = P_a^b(f) - N_a^b(f)$.*

Proof. The first statement is clear since $|f(x)| = f(x)^+ + f(x)^-$. Now for any parition P we have that $\sum_{i=1}^k [f(x_i) - f(x_{i-1})]^+ = \sum_{i=1}^k [f(x_i) - f(x_{i-1})]^- + f(b) - f(a)$ (write $f = f^+ - f^-$ and the sum telescopes). Taking supremums gives $P_a^b(f) - N_a^b(F) = f(b) - f(a)$. //

Theorem 9.6. *f is a function of bounded variation on $[a, b]$ iff f is the difference of two monotone functions on $[a, b]$.*

Proof. Let f be of bounded variation, and define $g(x) = P_a^x(f) + f(a)$ and $h(x) = N_a^x(f)$. Then $g(x) - h(x) = P_a^x(f) - N_a^x(f) + f(a) = f(x) - f(a) + f(a) = f(x)$. Likewise $g(x)$ and $h(x)$ are monotone increasing functions. Conversly let $f = g - h$. Then for any partition P we have $\sum_{i=1}^k |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^k [g(x_i) - g(x_{i-1}] + \sum_{i=1}^k [g(x_i) - g(x_{i-1}] = g(b) - g(a) + h(b) - h(a) \leq \infty$. Taking supremums we are done. //

Incidentially we have that $f(x) = P_a^x(f) - N_a^x(f) + f(a)$ which are both increasing functions we will see how important this is in the next section.

9.2. **Differentiation.** Since not all functions are differentiable, sometimes to extend results we can talk only about left and right handed derivatives, or since derivatives are limits, either about the limsup or liminf in place of the derivative. These quantities are known as derivates:

Definition 9.7. *For a function $f(x)$ we have the following:*

(1) $D^+ f(x) = \overline{\lim_{h \to 0^+}} \frac{f(x+h) - f(x)}{h}$
(2) $D^- f(x) = \overline{\lim_{h \to 0^+}} \frac{f(x) - f(x-h)}{h}$
(3) $D_+ f(x) = \underline{\lim_{h \to 0^+}} \frac{f(x+h) - f(x)}{h}$
(4) $D_- f(x) = \underline{\lim_{h \to 0^+}} \frac{f(x) - f(x-h)}{h}$

We have that $D^+f(x) \geq D_+f(x)$ and $D^-f(x) \geq D_-f(x)$, also we say that f is differentiable provided $d^+f(x) = D_+f(x) = D^-f(x) = D_-f(x)$. f has a right hand derivative when $D^+f(x) = D_+f(x)$ and similarly for left hand derivatives. The following proposition is proved in the homework section:

Proposition 9.8. *If f is continuous on $[a,b]$ and one of its derivates is everywhere non-negative then f is non-decreasing on $[a,b]$.*

We have then the first weak generalization of fundemental theorem:

Lemma 9.9. *Let f be an monotone differentable a.e. real valued function on $[a,b]$. Then its almost everywhere derivative f' is measurable and $\int_a^b f'(x)dm \leq f(b) - f(a)$.*

Proof. Let $g(x) = f'(x)$, and $g_n(x) = n[f(x+\frac{1}{n}) - f(x)]$, then $\{g_n(x)\} \to g$. Redefine f so that $f(x) = f(b)$ for $x \geq b$. Then g_n is measurable and since f is monotone we have $g_n \geq 0$. This gives by Fatou, $\int_a^b g \leq \underline{\lim} \int_a^b g_n = \underline{\lim}\, n \int_a^b (f(x + \frac{1}{n}) - f(x))dm = \underline{\lim}\, n(\int_b^{b+\frac{1}{n}} fdm - \int_a^{a+\frac{1}{n}} fdm) = \underline{\lim}(f(b) - n\int_a^{a+\frac{1}{n}} fdm \leq f(b) - f(a)$. //

This condition is not so restrictive as it can be shown that

Theorem 9.10. *Let f be a increasing function on $[a,b]$. Then f is differentable a.e. on $[a,b]$.*

Proof. We show that f is differentable almost every by showing the the derivates all agree on the complement of a measure zero set. Without loss of generality consider $D^+f(x)$ and $D_-f(x)$. Let $E_{u,v} = \{x | D^+f(x) > u > v > D_-f(x)\}$ for rational $u > v$. Then it suffices to show that $m^*(E_{u,v}) = 0$. Let $\epsilon > 0$, and choose a open set O such that $mO < m^*(E_{u,v}) + \epsilon$. For each $x \in E_{u,v}$ we can find a interval $[x-h, x] \subset O$ such that $f(x) - f(x-h) < vh$. Then the collection $\{[x-h,x]\}$ forms a Vitali covering of $E_{u,v}$. Let $I_1, \ldots, I_n \in \{[x-h,x]\}$ such that $\sum l(I_j) > m^*(E_{u,v}) - \epsilon$. Then we have that over these intervals we obtain $\sum_{j=1}^N [f(x_j) - f(x_j - h_j)] < v\sum_{j=1}^N h_j < vm(0) < v(m^*(E_{u,v}) + \epsilon)$. Let $A = E_{u,v} \cap \bigcup_{j=1}^N I_j$. Then for each $y \in A$ we can find $(y, y+k)$ such that $(y, y+k) \subset I_j$ for some j and $f(y+k) - f(y) > uk$. Then $\{(y, y+k)\}$ is also a Vitali covering. Therfore we can find $J_1, \ldots, J_M$ such that $\sum l(J_j) > m^*(E_{u,v}) - 2\epsilon$. Then summing over these intervals we have $\sum_{i=1}^M f(y_i + k_i) - f(y_i) > u\sum k_i > u(m^*(E_{u,v}) - 2\epsilon)$. Each interval J_i is contained in some I_j, and so if we sum over those intervals $J_i \subset I_j$ then we have $\sum f(y_i + k_i) - f(y_i) \leq f(x_j) - f(x_j - h_j)$ since f is increasing. So we have $\sum_{j=1}^N f(x_j) - f(x_j - h_j) \geq \sum_{i=1}^M f(y_i + k_i) - f(y_i)$ which gives that $v(m^*(E_{u,v}) + \epsilon) > u(m^*(E_{u,v}) - 2\epsilon)$ which gives that $vm^*(E_{u,v}) \geq um^*(E_{u,v})$. But we had that $u > v$ so this gives that $m^*(E_{u,v}) = 0$, and so f is differentiable a.e. //

Corollary 9.11. *If f is a function of bounded variation on $[a,b]$ then f' exists almost everywhere on $[a,b]$.*

9.3. **Fundemental Theorem of Calculus.** So we have some generalization of the first fundemental theorem. Now we approach the second, namly consider the function $F(x) = \int_a^x f(t)dm$. We will find that for integrable function the fundemental theorem still holds. We will then explore an important class of functions in the next section that will give a necessary and sufficent condition for when the fundemental theorem holds.

Lemma 9.12. *Let f be an integrable function on $[a, b]$, then $F(x) = \int_a^x f(t)dm$ is a continuous function of bounded variation on $[a, b]$*

Proof. We know already that the integral is continuous as follows by the monotone convergence theorem. Let P be a partition of $[a, b]$. Then $\sum |F(x_i) - F(X_{i-1}) = \sum |\int_{x_{i-1}}^{x_i} f(t)dm| \leq \sum \int_{x_{i-1}}^{x_i} |f(t)|dm = \int_a^b |f(t)|dm$ so we have $T_a^b(F) \leq \int_a^b |f(t)|dm < \infty$ as $f \in L^1$. //

Lemma 9.13. *Let f be an integrable function on $[a, b]$, such that $\int_a^x f(t)dm = 0$ for $x \in [a, b]$. Then $f(t) = 0$ a.e. on $[a, b]$.*

Proof. Let $E = \{x|f(x) > 0\}$, and $m(E) > 0$. We have that $\int_{a'}^{b'} f = \int_a^{b'} fdm - \int_a^{a'} fdm = 0$ which gives that for any $U \subset [a, b]$ open, then $\int_U fdm = 0$ which gives by complements that $\int_F fdm$ for any closed $F \subset [a, b]$. Let $\epsilon > 0$ and δ such that $m(A) < \delta$ then $\int_A f < \epsilon$. Let F be a closed set such that $m(E \setminus F) < \delta$ and so we have that $\int_E f = \int_F f + \int_{E \setminus F} f$ which gives that $\int_E f = \int_{E \setminus F} f$ and since $m(E \setminus F) < \delta$ we have that $\int_E fdm < \epsilon$. Since ϵ was arbitrary we have that $\int_E f = 0$ a contradiction. //

We start with the following attempts at generalization

Lemma 9.14. *Let f be a bounded measurable function on $[a, b]$, and $F(x) = \int_a^x f(t)dm$ then $F'(x) = f(x)$ a.e. on $[a, b]$*

Proof. First note that F is of bounded variation, and so $F'(x)$ exists a.e. Let $|f| \leq K$. Let $f_n = n[f(x + \frac{1}{n}) - f(x)] = n \int_x^{x+\frac{1}{n}} fdm$. Then we have that $f_n \to F'$ and $|f_n \leq K$ and so we have that $\int_a^c f_n \to \int_a^c F'$ for each $c \in [a, b]$, by the convergence theorems (LDCT or Bounded convergence). This gives then that $\int_a^c (F'(x) - f(x))dm = 0$ for each $c \in [a, b]$ so $F'(x) = f(x)a.e.$ by the previous lemma. //

We now extend this to integrable functions.

Theorem 9.15. *Let f be integrable on $[a, b]$ and $F(x) = F(a) + \int_a^x f(t)dm$ then $F'(x) = f(x)$ a.e. on $[a, b]$.*

Proof. Without loss of generality we have that $f \geq 0$. Let $f_n = max(f(x), n)$ then f_n is bounded and non-negative, so let $G_n = \int_a^x (f - f_n)dm$. Then G is increasing and differentiable almost everywhere such that $G_n'(x) \geq 0$. Now we have that $F(x) = \int_a^x f = \int_a^x f_n + \int_a^x (f - f_n) = \int_a^x f_n + G_n(x)$. So we have by the previous lemma $\frac{d}{dx} \int_a^x f_n = f_n(x)$ a.e. Now we have $F'(x) = f_n + G_n'(x)$ a.e. which gives that $F'(x) \geq f_n$ for each n and thus $F'(x) \geq f$ a.e. This gives then that $\int_a^b F' \geq \int_a^b f = F(b) - F(a)$ so we have then that $F' = f$ a.e. //

9.4. **Absolute Continuity of functions.**

Definition 9.16. *A function f on $[a, b]$ is said to be absolutly continuous (a.cts) provided that $\forall \epsilon > 0$, $\exists \delta > 0$ such that for non-overlapping intervals (may share endpoints) $\{(x_i, y_i)\}$ such that $\sum_{i=1}^n |y_i - x_i| < \delta$ then $\sum_{i=1}^n |f(y_i) - f(x_i)| < \epsilon$.*

We note that a.cts functions are uniformly so (Let $n = 1$ in the above definition), and we will show that they are of bounded variation, hence differntiable almost everywhere.

Lemma 9.17. *Let f be a.cts on $[a, b]$. Then f is of bounded variation.*

Proof. Let f be a.cts on $[a, b]$. Let δ be such that when $\sum_{i=1}^{n} |y_i - x_i| < \delta$ for $\{(x_i, y_i)\}$, then $\sum_{i=1}^{n} |f(y_i) - f(x_i)| < 1$. Then any partition of $[a, b]$ has a refinement that can be split into K intervals each of total length less than δ where $K = \lfloor 1 + \frac{b-a}{\delta} \rfloor$. Then we have that $T_a^b(f) \leq K$. //

We have as a quick corollary that absolutely continuous functions are differentiable almost everywhere.

Lemma 9.18. *Let f be a.cts on $[a, b]$ such that $f'(x) = 0$ a.e., then f is constant.*

Proof. Let f be a.cts on $[a, b]$ such that $f'(x) = 0$ a.e. Let $c \in [a, b]$ Then we want ot have htat $f(c) = f(a)$. Let $\epsilon, \eta > 0$. Define $E = \{x | f'(x) = 0 \text{ for } x \in (a, c)\}$. Then we have that $m(E) = c - a$. Let $I = \{[x, x + h]\}$ be such that $[x, x + h] \subset [a, c]$, and $|f(x + h) - f(x)| < \eta * h$. Then I is a Vitali cover. Let δ be such that $\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \epsilon$ for $\sum_{i=1}^{n} |y_i - x_i| < \delta$. By the Vitali lemma we have that finitely many $[x_i, y_i]_{i=1}^{n}$ such that $m(E \setminus \bigcup_i [x_i, y_i]) < \delta$, then $\sum |x_{i+1} - y_i| < \delta$. This gives then that $\sum |f(y_i) - f(x_i)| \leq \eta \sum |y_i - x_i| < \eta(c - a)$ and $|f(c) - f(a)| = |\sum f(x_{i+1} - f(y_i)| + \sum |f(y_i) - f(x_i)| \leq \epsilon + \eta(c - a) \Rightarrow f(c) - f(a) = 0$. //

We now prove the last version of the fundemental theorem, essentially stating that the fundemental theorem holds only for a.cts functions.

Theorem 9.19. *A function F is an indefinite integral iff it is absolutely continuous.*

Proof. If F is an indefinite integral, then it is a.cts by the continuity of the integral. Conversly let F be a.cts on $[a, b]$, then F is of bounded variation and so we can write $F = F_1 - F_2$ where F_1, F_2 are monotone increasing functions. So $F'(x)$ exists almost everywhere, and we have that $|F'(x)| \leq F_1'(x) + F_2'(x)$. This gives that $\int |F'(x)| \leq F_1(b) + F_2(b) - F_1(a) - F_2(a)$. Let $G(x) = \int_a^x F'(t)dm$ then G is a.cts and $F' = G'$ a.e. This gives then that $F = G + K$ for some constant K, then $F = \int_a^x F'(t)dm$. //

In general telling whether or not a function is of bounded variation or not can be difficult, however functions with bounded derivatives tend to behave very nicely.

Theorem 9.20. *Given a function continuous on $[a, b]$ and differentiable on (a, b) with bounded derivative $|f'(x)| \leq M$ on (a, b) then f is of bounded variation f is absolutely continuous*

Proof. Naturally bounded variation follows from our function being absolutely continuous. So we consider $\sum_{i=1}^{n} |f(x_i) - f(y_i)| = \sum_{i=1}^{n} |f'(c_i)||x_i - y_i|$ by the mean value theorem wher $c_i \in (x_i - y_i)$. Then we have that $\sum_{i=1}^{n} |f(x_i) - f(y_i)| \leq M \sum_{i=1}^{n} |x_i - y_i|$. So if we desire $\sum_{i=1}^{n} |f(x_i) - f(y_i)| < \epsilon$ then we would take $\sum_{i=1}^{n} |x_i - y_i| < \frac{\epsilon}{M}$ //

10. Convex Functions

We pause to give a all be it short description of a very special class of functions.

Definition 10.1. *A function $\varphi : (a, b) \to \mathbb{R}$ will be called convex provided $\forall x, y \in (a, b)$ and $0 \le \lambda \le 1$ then $\varphi(\lambda x + (1 - \lambda)y) \le \lambda\varphi(x) + (1 - \lambda)\varphi(y)$.*

Graphically we have that a given $0 \le \lambda \le 1$ corresponds to a point $a \le t \le b$. Then the convex property is equivalent to $(t, \varphi(t))$ lying above the chord connecting $(a, \varphi(a))$ and $(b, \varphi(b))$. That is if $a < s < t < u < b$ then we have that

$$\frac{\varphi(t) - \varphi(s)}{t - s} \le \frac{\varphi(u) - \varphi(t)}{u - t}$$

We now will prove two key results about convex functions. The first is a classification of its continuity and differentiability properties. The second is a key inequality due to Jensen which will be an important cog in our study of linear spaces.

Proposition 10.2. *Given φ a convex function on (a, b), then φ is absolutely so on any closed subinterval, continuous on (a, b) and is differentiable almost everywhere.*

Proof. Let $[c, d] \subset (a, b)$. Then we have that $\frac{\varphi(c)-\varphi(a)}{c-a} \le \frac{\varphi(y)-\varphi(x)}{y-x} \le \frac{\varphi(b)-\varphi(d)}{b-d}$ for $x, y \in [c, d]$. Thus we have $|\varphi(y) - \varphi(x)| \le M|x - y|$. and so φ is absolutely continuous on $[c, d]$. Let $x_0 \in (a, b)$. Then we have that $\frac{\varphi(x)-\varphi(x_0)}{x-x_0}$ is an increasing function and so must be differentiable hence continuous off set of zero measure. This gives that the left and right hand limits exists and agree. //

The next proposition is a essential inequality

Proposition 10.3. *(Jensen) Let $(X, \mathcal{F}, \mu)$ be a space such that $\mu(X) = 1$. Let φ be a convex function on (a, b). Then for any integrable function f such that $a < f(x) < b$ we have that $\varphi(\int f d\mu) \le \int(\varphi(f))d\mu$.*

Proof. Let $\alpha = \int f d\mu$. Then $a < \alpha < b$. Let $\beta = \sup\limits_{a<s<\alpha} \frac{\varphi(\alpha)-\varphi(s)}{\alpha-s}$. Then we have that $\varphi(s) \ge \varphi(\alpha) + \beta(s - \alpha)$ where $a < s < b$. This gives that for each $x \in X$ we have that $\varphi(f(x)) - \varphi(\alpha) - \beta(f(x) - \alpha) \ge 0$. Now since φ is convex, whence continuous we have that $\varphi(f)$ is measurable, and so we can integrate to obtain $\int \varphi(f)d\mu - \int \varphi(\alpha)d\mu - \beta(\int f d\mu - \alpha) \ge 0$ which gives that $\int \varphi(f)d\mu \ge \varphi(\alpha) = \varphi(\int f d\mu)$.

//

We exhibit some of these inequalities, for example e^x, $log(x)$, and x^p are both convex functions where $p \ge 1$. We have then by Jensen's inequality that $\int e^f \le e^{\int f}$, $\int log(f) \le log(\int f)$ and $(\int f)^p \le \int f^p$.

11. L^p spaces

We begin our discussion of L^p linear spaces by defining a useful arithmetical relationship.

Definition 11.1. *Let $p \in [1, \infty]$ then a extended real number q such that $\frac{1}{q} + \frac{1}{p} = 1$ is called a conjugate exponent of p.*

This relationship will be fantastically useful. Before we jump into the full description of L^p spaces, we will need one more useful lemma.

Lemma 11.2. *For $p \in (0,\infty)$ and a,b real numbers, then then we have that $(a+b)^p \leq 2^p(a^p+b^p)$.*

Proof. Since $a+b \leq 2\max(a,b)$ we have then that $(a+b)^p \leq 2^p \max(a^p,b^p)$ and so we have that $(a+b)^p \leq 2^p(a^p+b^p)$. //

Now we will describe our space. Fix a space $(X,\mathcal{F},\mu)$ and a value $p \in (0,\infty)$. Then for a given measurable function f we can examine $\int |f|^p$. Naturally we can have this quantity be infinite or finite.

First consider the collection $\{f|$ f is measurable and $\int |f|^p < \infty\}$.

Then we note that for $c \in \mathbb{R}$ and f,g in the collection then cf and $f+g$ are both in the collection as $\int |f+g|^p \leq \int 2^p(|f|^p+|g|^p)$. This makes the collection a real vector space. Further we can propose a norm on this collection by $||f||_p = (\int |f|^p d\mu)^{\frac{1}{p}}$. However if this is to be a norm we had better have $||f||_p = 0$ and we don't have this necessarily as when $f=0 \quad \mu-$a.e then $\int |f|^p = 0$. So we make an equivalence relation $f \sim g$ provided $f = g \quad \mu-$a.e. Under this equivalence we do have that $||f||_p$ gives an honest norm (we will show this momentarily), and so we will define:

Definition 11.3. *Given a space $(X,\mathcal{F},\mu)$ we define for $p \in [1,\infty)$ $L^p(\mu) = \{f|$ f is measurable and $\int |f|^p < \infty\}/\sim$ where $f \sim g$ if and only if $f = g \quad \mu-a.e.$*

In order for this to be a normed linear space we must show that

(1) $||f+g||_p \leq ||f||_p + ||g||_p$
(2) $||f||_p = 0$ iff $f = 0$
(3) $||cf||_p = |c|||f||_p$

We have already shown that $||f||_p = 0$ iff $f = 0$ where this equivalence is up to $\mu-$a.e. as described. We will show the other two properties later.

First we will introduce a very realted normed linear space. In order to do this we will talk about functions being bounded almost everywhere and we will denote the smallest such bound by the following definition

Definition 11.4. *For a given space $(X,\mathcal{F},\mu)$ and a $\mu-$measurable function. Then we define $||f||_\infty = \inf(M|\mu(t|f(t) > M) = 0)$ that is $||f||_\infty = \inf(M||f| \leq M \quad \mu-$a.e.). We will call $||f||_\infty$ the essential supremum (or ess sup) of f, and if $||f||_\infty < \infty$ then we will call f essentially bounded.*

Notice that the space L^1 is exactly the space of $\mu-$integrable functions. We can also construct then the linear space $L^\infty = \{f|||f||_\infty < \infty\}$, under the same equivalence as before.

Now we will begin working through the details of showing that the L^p spaces (here $p \in [1,\infty]$) are normed linear spaces. This structure will allow use to discuse convergence of sequences in L^p and duality. We will gain some useful inequalities for estimating the size of integrals when dealing with functions in a particular L^p.

Lemma 11.5. *(Young) For given real numbers $\alpha,\beta \geq 0$ then we have that $\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$ where $p \in (1,\infty)$ and q is the conjugate. Equality happends when $\alpha^p = \beta^q$.*

Proof. If $\alpha = 0$ or $\beta = 0$ then we are done, so assume $\alpha > 0$ and $\beta > 0$. Let $\varphi(t) = \frac{\alpha^p}{p} + \frac{t^q}{q} - \alpha t$, for $t > 0$. Now we will minimize $\varphi(t)$. We differentiate $\varphi'(t) = t^{q-1} - \alpha$ and $\varphi''(t) = (q-1)t^(q-2) > 0$ for $t > 0$ so our function is convex. Next we find the minimum by finding the root $\alpha^{\frac{1}{q-1}}$ to our first derivative. Now since $\frac{1}{p} + \frac{1}{q} = 1$ we have that $p+q = pq$ and $\frac{1}{q-1} = \frac{p}{q}$ so we have that $\alpha^{\frac{1}{q-1}}$ then since

$\varphi(\alpha^{\frac{1}{q-1}}) = \frac{\alpha^p}{p} + \frac{\alpha^{\frac{pq}{q}}}{q} - \alpha^{\frac{p}{q}+1} = \frac{\alpha^p}{p} + \frac{\alpha^p}{q} - \alpha^{\frac{p}{q}+1} = \alpha^p(\frac{1}{p}+\frac{1}{q}) - \alpha^{\frac{p+q}{q}} = \alpha^p - \alpha^{\frac{pq}{q}} = 0$, hence we have a minimal value for $\varphi(t)$ of 0, which gives that $\frac{\alpha^p}{p} + \frac{t^q}{q} \geq \alpha t$. Likewise we attain our minimum at $\alpha^{\frac{1}{q-1}} = \alpha^{\frac{p}{q}} = \beta$ which gives equivalence when $\alpha^p = \beta^q$. //

Next we prove an estimate theorem for integrals.

Proposition 11.6. *(Hölder) For a given space $(X, \mathcal{F}, \mu)$, let p, q be conjugate indicies. Then for $f \in L^p$ and $g \int L^q$ such that $f, g \geq 0$, then we have that $\int |fg| d\mu \leq ||f||_p ||g||_q$, and equality iff $c|f|^p = d|g|^q$ for some $c, d \in \mathbb{R}$.*

Proof. if $p = 1$ and $q = \infty$ then we are done, so assume that $p \in (1, \infty)$. If either $||f||_p = 0$ or $||g||_q = 0$ then either $f = 0 \quad \mu-$a.e. or $g = 0 \quad \mu-$a.e. and so we would have $fg = 0 \quad \mu-$a.e. and $\int |fg| d\mu = 0$, and the inequality is true. So now assume $||f||_p \neq 0$ and $||g||_q \neq 0$. Now, consider $f_0 = \frac{f}{||f||_p}$ and $g_0 = \frac{g}{||g||_q}$. Then for each x we have that $|f_0(x)||g_0(x)| \leq \frac{|f_0(x)|^p}{p} + \frac{|g_0(x)|^q}{q}$ by Young's inequality. Integrating we have $\int |f_0 g_0| d\mu \leq \int \frac{|f_0|^p}{p} + \int \frac{|g_0|^q}{q} = \frac{||f_0||_p^p}{p} + \frac{||g_0||_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1$. Now this gives that $\int |f_0 g_0| d\mu = \frac{\int |fg| d\mu}{||f||_p ||g||_q} \leq 1$ and the inequality is solved. We have equivalence when Young's inequality is equality which gives that we would have $|f_0(x)||g_0(x)| = \frac{|f_0(x)|^p}{p} + \frac{|g_0(x)|^q}{q}$ which again by Young's lemma occurs when $|f_0(x)|^p = |g_0(x)|^q$ which happens when $||g||_q |f|^p = ||f||_p |g|^q$. //

We now give the final basic inequality for the L^p spaces.

Proposition 11.7. *(Minkowski) For a given space $(X, \mathcal{F}, \mu)$, let $f, g \in L^p(\mu)$ for $p \in [1, \infty]$. Then $||f + g||_p \leq ||f||_p + ||g||_p$.*

Proof. We have that for $p = 1$ this is the triangle inequality and for $p = \infty$ we can take the larger essential sup and we would be done, so assume $p \in (1, \infty)$. Now without loss of generality assume $||f + g||_p \neq 0$. Then $||f + g||_p^p = \int |f + g|^p = \int |f + g||f + g|^{p-1} \leq \int |f|(|f + g|^{p-1}) + |g|(|f + g|^{p-1}) = \int |f|(|f + g|^{p-1} + \int |g|(|f + g|^{p-1})$. Now let q be the conjugate index of p. By lemma we have that $|f + g|^p \leq 2^p(|f|^p + |g|^p)$ and so since $f, g \in L^p$ we have that $f + g \in L^p$. Then we have that $(p-1)q = pq - q = p$ and so $\int |f + g|^{(p-1)^q} = \int |f + g|^p \leq 2^p(\int |f|^p + \int |g|^p) < \infty$ and so we have that $|f + g|^{p-1} \in L^q(\mu)$ hence we can use Hölder to obatain $||f + g||_p^p \leq \int |f|(|f + g|^{p-1} + \int |g|(|f + g|^{p-1}) \leq (\int |f|^p)^{\frac{1}{p}} \int (|f + g|^{(p-1)^q})^{\frac{1}{q}} + (\int |g|^p)^{\frac{1}{p}} \int (|f + g|^{(p-1)^q})^{\frac{1}{q}} = (||f||_p + ||g||_p)(\int |f + g|^p)^{\frac{1}{q}} = (||f||_p + ||g||_p)(\int |f + g|^p)^{\frac{1}{q}\frac{p}{q}} = (||f||_p + ||g||_p)(\int |f + g|^p)^{\frac{1}{q}\frac{p}{q}} = (||f||_p + ||g||_p)(||f + g||_p^{\frac{p}{q}})$. So we have that $||f + g||_p^p \leq (||f||_p + ||g||_p)(||f + g||_p^{\frac{p}{q}})$ or $\frac{||f+g||_p^p}{||f+g||_p^{\frac{p}{q}}} \leq ||f||_p + ||g||_p$ and since $\frac{||f+g||_p^p}{||f+g||_p^{\frac{p}{q}}} = ||f + g||_p^{p - \frac{p}{q}} = ||f + g||_p$ so the inequality is solved. //

The fruit of this is that we have for $p \in [1, \infty)$ then

(1) $||f + g||_p \leq ||f||_p + ||g||_p$
(2) $||f||_p = 0$ iff $f = 0$
(3) $||cf||_p = |c|||f||_p$

. (1) is just the last inequality, (2) we showed earlier and (3) is true ($\int |c||f|^p = |c| \int |f|^p$) and so we have that L^p is a normed linear space. Hence we can now discuss attributes of this.

Now since the space L^p is a normed linear space, it makes sense to talk about convergence in this norm, that is $f_n \to f$ in mean of order p provided $||f_n - f||_p \to 0$. It is fortuitus that our space is actually complete with respect to this norm. To prove this we use the following lemma.

Lemma 11.8. *A normed linear space X is complete if and only if ever absolutely summable series is summable*

Proof. Let X be a complete normed linear space. Then given a absolutely summable series $\sum f_n$ such that $\sum ||f_n|| = M < \infty$. Let $s_n = \sum_{i=1}^n f_n$ be the partial sum. Now we can for any $\epsilon > 0$ we can find N such that for $\forall n \geq N$ such that $\sum_{n=N}^{\infty} ||f_n|| < \epsilon$. So for $n \geq m \geq N$ then $||s_n - s_m|| = || \sum_{i=m}^n f_i|| \leq \sum_{i=m}^n ||f_i|| \leq \sum_{i=N}^{\infty} ||f_i|| < \epsilon$. Since our sequence is cauchy and in our complete space must converge.

Now assume absolutely summable series is summable. Let $\{f_n\}$ be any Cauchy. Then for each k we can find n_k such that $||f_n - f_m|| \leq 2^{-k}$ for $n, m \geq n_k$. Without loss of generality choose $n_k < n_{k+1} < \ldots$. Then $\{f_{n_k}\}$ is a subsequence. Let $g_1 = f_{n_1}$ and $g_k = f_{n_k} - f_{n_{k-1}}$ for $k > 1$. Then we get a sequence $\{g_k\}$ whose such that $\sum_{i=1}^k g_k = \sum_{i=1}^k f_{n_k} - f_{n_{k-1}} = f_{n_k}$. So $||g_k|| = 2^{-k+1}$ for $k > 1$ and so $\sum ||g_k|| = ||g_1|| + \sum 2^{-k+1} = ||g_1|| + 1$. So the series $\sum g_k$ is absolutely summable and so we can find f such that $f_{n_k} \to f$. Now since f_n is cauchy so let $\epsilon > 0$ and let N $||f_n - f_m|| < \frac{\epsilon}{2}$ for $n, m \geq N$. Since $f_{n_k} \to f$ then there is K such that for $k \geq K$, we have $||f_{n_k} - f|| < \epsilon\epsilon 2$. Let k be such that $k > K$ and $n_k > N$. Then $||f_n - f|| \leq ||f_n - f_{n_k}|| + ||f_{n_k} - f|| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ so $f_n \to f$. //

Now we will use this to show that L^p is complete.

Theorem 11.9. *(Riesz-Fischer) L^p is complete.*

Proof. It suffices to show that any absolutely summable series in L^p is summable. Give $\sum_{k=1}^{\infty} f_k$ such that $\{f_k\} \subset L^p$, and $\sum_{k=1}^{\infty} ||f_k||_p = M < \infty$, then we will show that $\sum_{k=1}^{\infty} f_k = g \in L^p$ is summable. Consider $h_n = \sum_{k=1}^n |f_k|$. Let h be a measurable function such that $h_n \to h$ $\mu-$a.e. We have that $||h_n||_p \leq \sum_{k=1}^n ||f_k||_p < M$ by Minkowski's inequality. This gives then that $||h_n||_p^p = \int h_n^p \leq M^p$ and so we have $\int h^p \leq M^p$ which gives that h is finite $\mu-$a.e. Where $h(x) < \infty$ we have that $g(x) = \sum_{k=1}^{\infty} f_k(x)$. Since $\sum_{k=1}^{\infty} f_k$ is absolutely summable we have that $\sum_{k=1}^{\infty} |f_k(x)| < \infty$ and so we have that $\sum_{k=1}^{\infty} f_k(x)$ is a real number. Let $g(x) = 0$ where $h(x) = \infty$. Now let $g_n = \sum_{k=1}^{\infty}$, then we have that $|g_n| \leq |h_n| \leq h$ which gives that $|g| \leq h$ and so $g \in L^p$. Now we have that $g_n \nearrow g$ $\mu-$a.e., and since $|g_n - g|^p \leq 2^{p+1}|g|^p$ we have that $\int |g_n - g|^p \to 0$ by dominated convergence, and so $\sum_{k=1}^{\infty} f_k = g$ is summable, whence L^p is complete. //

Now that we have convergence in L^p we have the following fantastic equivalence.

Proposition 11.10. *Let $(X, \mathcal{F}, \mu)$, and consider $L^p(\mu)$. Then given a sequence $\{f_n\} \subset L^p(\mu)$ such that $f_n \to f \in L^p(\mu)$. Then we have that $||f_n||_p \to ||f||_p$ iff $||f_n - f||_p \to 0$.*

Proof. Since $||f + g||_p \leq ||f||_p + ||g||_p$, we have that $||f - g||_p \leq ||f||_p - ||g||_p$ then given that $||f_n - f||_p \to 0$ we have then that $||f_n||_p - ||f||_p \leq ||f_n - f||_p \to$

0. Conversly assume $||f_n||_p \to ||f||_p$. Consider $g_n = |f_n - f|^p$. Then $g_n \to 0$. Likewise $g_n \leq 2^p(|f_n|^p + |f|^p) = h_n$. Then we have that $h_n \to 2^{p+1}|f|^p$. Likewise $\int h_n = 2^p(\int |f_n|^p + \int |f|^p) \to 2^{p+1}|f|^p = \int h$ by hypothesis. Therefore by lebesgue dominated covergence theorem $\int g_n = \int g = \int 0 = 0$. //

One powerful aspect about L^p functions is that they are easily approximated.

Proposition 11.11. *For $(X, \mathcal{F}, \mu)$ a space, then Simple functions and continuous functions of compact support are dense in $L^p(\mu)$.*

Proof. Given $(X, \mathcal{F}, \mu)$ a space, and $f \in L^p$, then we have a sequence φ_n of simple functions such that $\lim \varphi_n = f$, and $\varphi_n \leq f$. Since $f \in L^p$ this gives that $\varphi_n \in L^p$, and $|\varphi_n - f|^p = |f - \varphi_n|^p \leq |f|^p$ we have that $||\varphi_n - f||_p = (\int |\varphi_n - f|^p)^{\frac{1}{p}} \to 0$ by the dominated convergence theorem, thus simple functions are dense in L^p. Given a continuous function g with compact support, then given $\epsilon > 0$ we can find a simple function φ such that for $E = \{x | g \neq \varphi\}$ we have $\mu(E) < \epsilon$ and $|g| \leq \varphi$. This gives that $||g - \varphi||_p = (\int |g - \varphi|^p)^{\frac{1}{p}} = (\int_E |g - \varphi|^p + \int_{E^c} |g - \varphi|^p))^{\frac{1}{p}} \leq (\int_{E^c} |g| + ||\varphi||_\infty))^{\frac{1}{p}} = \mu(E^c))^{\frac{1}{p}} 2||\varphi||_\infty = \epsilon^{\frac{1}{p}} 2||\varphi||_\infty$, since ϵ was arbitrary the result follows. //

Fixing a space $(X, \mathcal{F}, \mu)$, One may ask the natural question as to when given a function f in $L^p(\mu)$ can we say that $f \in L^r$ for $p < r$ or $r < p$. The solution tends to lend itself well to picture. Fixing $f \in L^p$ lowering the power brings our function closer to the constant function 1. This would imply that if we would like $f \in L^r$ then we would need 1 to be integrable, hence a finite space. To get the opposite direction, we note that any spot where our function is blowing up will only blow up more, hence we need these uncontrolled blow ups to happen on a negligable set, hence we need our functions to be essentially bounded. We present this as follows:

Proposition 11.12. *Given $(X, \mathcal{F}, \mu)$, then for $f \in L^p$ we have that $f \in L^r$ for $r < p$ provided $\mu(X) < \infty$ and we have that $L^p \cap L^\infty \subset L^r$ for $r > p$.*

Proof. Fix a finite space $(X, \mathcal{F}, \mu)$, and $f \in L^p$. Then for $r < p$ we have that $\int |f|^r d\mu = \int_{f<1} |f|^r d\mu + \int_{f \geq 1} |f|^r d\mu \leq \int_{f<1} 1 d\mu + \int_{f \geq 1} |f|^p d\mu < \infty$, so $f \in L^r$

Now given any space $(X, \mathcal{F}, \mu)$, we have that for $f \in L^p \cap L^\infty$ then for $p < r$ we have that $\int |f|^r d\mu = \int_{f<1} |f|^r d\mu + \int_{f \geq 1} |f|^r d\mu \leq \int_{f<1} |f|^p d\mu + \int_{f \geq 1} ||f||_\infty^r d\mu < \infty$ as $f \in p$ gives that $\int_{f \geq 1} 1 d\mu \leq \int_{f \geq 1} |f|^p d\mu < \infty$ since $f^p \in L^1$. This concludes the result. //

After one has some idea about the nature of a linear space, one way to study the space further is to understand the linear functionals on the space (i.e., for a real vector space X these are the maps $F : X \to \mathbb{R}$). These maps algebraically form what is called the dual space X^*. Let us see what some bounded linear functionals look like on L^p. Given a function in L^q where q is the conjugate index to p, we can form $F : L^p \to \mathbb{R}$ by $F(f) = \int fg d\mu$, which would be bounded by Hölder's inequality. We have then that this is a bounded linear functional. We can estimate size of a bounded linear functional as follows:

Definition 11.13. *Given a space $(X, \mathcal{F}, \mu)$ and a bounded linear functional $F : L^p(\mu) \to \mathbb{R}$ then we define the norm of F to be $||F|| = \sup_{f \in L^p} |F(f)| = \sup_{||f||_p = 1} |F(f)|$.*

Alternatively we could have defined our norm $||F|| = \inf_{f \in L^p} \{M | |F(f)| \leq M||f||_p\}$. It is convenient to note that for a bounded linear functional then F is libschitz, uniformially continuous, and continuous exactly when it is continuous at a particular point. We have the following lemma to give some continuity conditions on our function.

Lemma 11.14. *Let F be a bounded linear functional. Then F is continuous*

Proof. We have that it suffices to show continuity at one point. For $\epsilon > 0$ let $||f||_p \leq \frac{||F||}{\epsilon}$ then we have that $||F(f)|| = \frac{|F(||f||_p f|}{||f||_p} \leq \frac{M}{||f||_p} = \epsilon$. //

It turns out that L^p spaces have very nice duals, namely L^q where q is the conjugate index for p. This result is called the Reisz Representation theorem.

Theorem 11.15. *(Reisz) Let $(X, \mathcal{F}, \mu)$ be a $\sigma-$finite measure space and $p \in [1, \infty)$. Let F be a bounded linear functional on $L^p(\mu)$. Then there is a unique element $g \in L^q$ where q is the conjugate index for p such that $F(f) = \int fgd\mu$. Also we have $||F|| = ||g||_q$.*

Proof. First consider $\mu(X) < \infty$. Then we have that for any measurable E, $\chi_E \in L^p$ for any p. We can then define $\lambda(E) = F(\chi_E)$. Given $E = \bigcup E_n$ all disjoint measurable, then we have htat $\lambda(\bigcup E_n) = F(\chi_{\bigcup E_n}) = F(\sum \chi_{E_n}) = \sum F(\chi_{E_n}) = \lambda(E_n)$. Further if we let $f = \sum(sgn(F(\chi_{E_n}))\chi_{E_n}$, then $\sum |\lambda(E_n)| = F(f) < \infty$ so the sum is absolutely convergent and we have that λ is a signed measure. Now by the Radon-Nikodym theorem we have that $\lambda(E) = \int_E gd\mu$ and since λ is finite we have that g is integrable. Further we have that $g \in L^q$ since $\frac{|F(\varphi)|}{||\varphi||_p} \leq ||F||$ which gives that $|F(\varphi)| \leq ||\varphi||_p ||F||$. Now we will show that g is in L^q. We can find a sequence $\varphi_n \nearrow |g|^q$. Let $\psi_n = (sgng)\psi_n^{\frac{1}{p}}$. Then we have that $||\psi_n||_p = (\int \varphi_n d\mu)^{\frac{1}{p}}$. Now $\varphi_n = |\varphi_n|^{\frac{1}{p}+\frac{1}{q}} = |\psi_n||\varphi_n|^{\frac{1}{q}} \leq |\varphi_n||g|$. So we have that $\int \varphi_n d\mu \leq \int \psi_n gd\mu \leq ||F||||\psi_n||_p = M(\int \varphi_n d\mu)^{\frac{1}{p}}$ so this gives that by Monotone Convergence $\int |g|^q d\mu = \int \varphi_n d\mu \leq M^q$, thus $g \in L^q(\mu)$. Let $G(f) = \int fgd\mu$. Then $G - F$ is a bounded linear functional and $F(\varphi) = \int \varphi gd\mu$ for any simple function φ by linearity, so we have $(F - G)(\varphi) = 0$ for simple functions. This gives that $(F - G)(f) = 0$ for $f \in L^p$ by lemma. Finally we have that $||F|| = ||G|| = ||g||_q$.

Now to extend our proof to the $\sigma-$finite case. Let $\{X_n\}$ be an increasing sequence of measurable sets of finite measure whose union is X. Then for each n we can find $g_n \in L^q$ non-negative such that g_n vanishes outside X_n, and $F(f) = \int fg_n d\mu$. We have also that $||g_n||_q \leq ||F||$. Let $g(x) = g_n(x)$ for $x \in X_n$. Since the g_n's are almost unique, that is $g_{n+1}(x) = g_n(x) \quad \mu-$a.e, then g is well defined and we have that by the monotone convergence theorem $\int |g|^q d\mu = \lim \int |g_n|^q d\mu \leq ||F||$. Let $f \in L^p$ and $f_n = f\chi_{X_n}$. Then $fg \in L^1$ by Hölder and since $|f_n g| \leq |fg|$ we have that $\int fgd\mu = \lim \int f_n gd\mu = \lim \int f_n g_n d\mu = \lim F(f_n) = F(f)$ by Lebesgue dominated convergence. //

We provide a counter example to show that $\sigma-$finiteness is needed for $p = 1$.

Example 11.16. *Let $X = \{a\}$ and $\mathcal{F} = \wp(X)$. Let $\mu(\emptyset) = 0$ and $\mu(X) = \infty$. Then $L^1(\mu) = \{f \equiv 0\}$ and $L^\infty(\mu) = \mathbb{R} = \{f \equiv \alpha\}$ so $||f||_\infty = |\alpha|$, and for any linear functional F we have $0 = F(f)$ but for any $g \in L^\infty$ then $\int fgd\mu = \infty \neq 0$*

One can also show that the $\sigma-$finiteness is not needed for $p > 1$.

12. Modes of convergence

This section deals with differing ideas of convergence and how they related to one another. We have the following modes that will be considered:

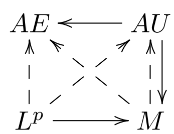
Definition 12.1. *Given a space $(X, \mathcal{F}, \mu)$ and a sequence $\{f_n\}$ of functions. Then we say that:*

1. *$\{f_n\} \to f \mu-$almost everywhere provided $\mu(\{x|f_n(x)$ does not converge to $f(x)\}) = 0$*
2. *$\{f_n\} \to f$ Almost uniform provided $\forall\epsilon > 0$ there is a set A such that $\mu(A) < \epsilon$ and $\{f_n\}$ converges uniformly to f on A^c.*
3. *$\{f_n\} \to f$ in measure provided $\forall\epsilon > 0$ we have that $\mu(\{x|\ |f_n(x) - f(x)| \geq \epsilon\}) \to 0$ as $n \to \infty$.*
4. *$\{f_n\} \to f$ in mean of order p provided $\int |f_n - f|^p \to 0$ as $n \to \infty$.*

We will for convenience denote these modes by the abbreviations AE for $\mu-$almost everywhwere, AU for almost uniform, M for measure and L^p for mean of order p. The question that will be asked now is when and under what circumstances does convergence in one mode imply convergence in the others. We will find that there are three basic hypotheses to cover and that each direction will be treated.

12.1. General Case.

For the general case we have the following diagram[5] describing the results:



In the diagram a solid arrow means convergence in the first mode gives converence in the second mode. A dotted arrow means that convergence in the first mode gives a subsequence that convergence in second mode. A lack of arrow shows that a counter example can be constructed. Note that arrows which follow by transitivity are also shown.

To prove this siutation we show the following directions, AU $\to$ M, AU $\to$ AE, M $\to L^p$, and that converging in M gives a subsequence which converges AU. The last result is known as Riesz lemma.

Lemma 12.2. *Given a space $(X, \mathcal{F}, \mu)$, Given a sequence $\{f_n\}$ which converges almost uniform, then it also converges $\mu-$a.e.*

Proof. Let $(X, \mathcal{F}, \mu)$ a space and $\{f_n\}$ a sequence which converges almost uniform to f. Then $\forall k \in \mathbb{N}$ we have a set A_k such that $\mu(A_k) < \frac{1}{k}$ such that $\{f_n\}$ converges uniformly to f on A_k^c. Let $A = \bigcap A_k$. Then $\mu(A) = 0$ and for each $x \in A^c = \bigcup A_k^c$ then $x \in A_k^c$ for some k which gives that $\{f_n\}$ converges uniformly on A_k^c, whence pointwise to f. Thus $\{f_n\}$ converges $\mu-$a.e. //

Lemma 12.3. *Given a space $(X, \mathcal{F}, \mu)$, Given a sequence $\{f_n\}$ which converges almost uniform , then it also converges in Measure.*

Proof. Let $(X, \mathcal{F}, \mu)$ a space and $\{f_n\}$ a sequence which converges almost uniform to f. Then we have that $\forall\delta > 0$ we can find A_δ such that $\{f_n\}$ converges uniformly on A_δ^c and $\mu(A_\delta) < \delta$. This gives that $\forall\epsilon > 0$ there is an N such that $\forall n \geq N$ we

[5]The latex for these diagrams were provided by Will Dicharry

have $|f_n(x) - f(x)| < \epsilon$, thus $\mu(\{x| \, |f_n(x) - f(x)| \geq \epsilon\}) \leq \mu(A_\delta) < \delta$. Therefore $\{f_n\}$ converges in meausure to f. //

Lemma 12.4. *Given a space $(X, \mathcal{F}, \mu)$, Given a sequence $\{f_n\}$ that converges in mean of order p then it converges in measure.*

Proof. Let $(X, \mathcal{F}, \mu)$ be a space, and $\{f_n\}$ be a sequence which converges in mean of order p. Given $\epsilon > 0$ let $E_n = \{x| \, |f_n(x) - f(x)| \geq \epsilon\}$ then we can write $\mu(E_n) = \int_{E_n} 1 d\mu$. We have also that $|f_n(x) - f(x)| \geq \epsilon \Rightarrow |f_n(x) - f(x)|^p \geq \epsilon^p$. Then $\epsilon^p \mu(E_n) = \int_{E_n} \epsilon^p \leq \int_{E_n} |f_n(x) - f(x)|^p \leq \int |f_n(x) - f(x)|^p \to 0$ as $n \to \infty$, therefore we have that $\epsilon^p \mu(E_n) \to 0$ as $n \to \infty$ which gives that $\mu(E_n) \to 0$ as $n \to \infty$. //

Lemma 12.5. *Given a space $(X, \mathcal{F}, \mu)$, then given a sequence $\{f_n\}$ which converges in Measure, then there is a subsquence $\{f_{n_k}\}$ such that converges almost uniform.*

Proof. Given a space $(X, \mathcal{F}, \mu)$ and $\{f_n\}$ a sequence which converges in measure. Then $\forall k \in \mathbb{N}$ we have that for $E_n = \{x| \, |f_n(x) - f(x)| \geq \frac{1}{k}\}$ then $\mu(E_n) \to 0$ as $n \to \infty$. Choose n_k such that $\forall n \geq n_k$ we have $\mu(E_n) < 2^{-k}$. Now let $A_k = \{x| \, |f_{n_k}(x) - f(x)| \geq \frac{1}{k}\}$ and let $G_j = \bigcup_{k=j}^{\infty} A_k$. Then for $x \in G_j^c \Rightarrow x \in A_k^c$ for $k \geq j$. This gives that $|f_{n_k}(x) - f(x)| < \frac{1}{k}$ for $k \geq j$. Now let $G = \bigcap G_j$. Then we have $\mu(G) \leq \mu(G_j) \leq \sum_{k=j}^{\infty} \mu(A_k) = \sum_{k=j}^{\infty} 2^{-k} = 2^{-j+1}$ for each j which gives that $\mu(G) = 0$. Now for $x \notin G$ we have $x \notin G_j$ for each j which gives that $|f_{n_k}(x) - f(x)| < \frac{1}{k}$ for each k. Let $\epsilon > 0$, choose k such that $\frac{1}{k} < \epsilon$. Then we have for $x \notin G$ we have for $n_l \geq n_k$, then $|f_{n_l}(x) - f(x)| < \frac{1}{k} < \epsilon$. Since n_k did not depend on x we have that the convergence is almost uniform. //

We have the following examples which offer counter examples for the rest of the directions.

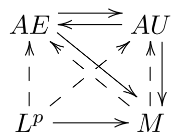
Example 12.6. *Consider $(\mathbb{R}, \mathcal{M}, m)$. Let $f_n = \chi_{[n,n+1]}$. Then we have that $f_n \to 0$, however $E_n = \{x| \, |f_n(x) - f(x)| \geq \frac{1}{2}\} = [n, n+1]$ and so $m([n, n+1]) = 1 \forall n$. This gives that convergence $\mu-$a.e. does not give convergence in measure and by transitivity we have that convergence $\mu-$a.e. cannot give convergence almost uniform.*

Example 12.7. *Consider $([0, \infty), \mathcal{M}, m)$. Let $f_n = n\chi_{[\frac{1}{n}, \frac{2}{n}]}$. Then again we have that $f_n \to 0$. However given $\epsilon > 0$ choose N such that $\frac{2}{\epsilon} < N$. Let $A = [0, \frac{2}{N}]$. Then $m(A) = \frac{2}{N} < \epsilon$. Now for any $x \in A^c$ we have that $x > \frac{2}{N} > \frac{2}{n}$ for $n \geq N$. Thus we have that for $n \geq N$ and $x \in A^c$ then $|f_n(x)| = 0$. This gives that the convergence is almost uniform. However consider $\int |f|^p = \int n^p \chi_{[\frac{1}{n}, \frac{2}{n}]} = n^p \cdot \frac{1}{n} = n^{p-1} \geq 1$ for $p \geq 1$. This gives that almost uniform convergence does not give convergence in mean of order p. Likewise by transitivity we also cannot have convergence in measure giving convergence in mean of order p nor convergence $\mu-$a.e. giving convergence in mean of order p.*

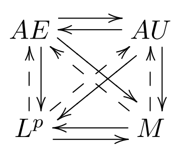
Example 12.8. *Consider $(\mathbb{R}, \mathcal{M}, m)$. Let $\{I_n\}$ be a enumeration of subintervals of the unit interval by: $I_1 = [0, 1], I_2 = [0, \frac{1}{2}], I_3 = [\frac{1}{2}, 1], I_3 = [0, \frac{1}{3}], I_4 = [\frac{1}{3}, \frac{2}{3}], I_5 = [\frac{2}{3}, 1], \ldots$. Let $f_n = \chi_{I_n}$. Then we have that f_n does not converge $\mu-$a.e. However for $n > \frac{m(m+1)}{2}$ we have that $\int f_n < \frac{1}{m}$ which gives that $\{f_n\}$ does converge in mean or order p to $f = 0$. Thus convergence in mean of order p does not give convergence*

$\mu-$a.e. This gives that by transitivity we have that convergence in measure cannot give convergence $\mu-$a.e., convergence in measure cannot give convergence almost uniform, and that convergence in mean or order p cannot give convergence almost uniform.

12.2. **Finite Case.** The finite case handels the added hypothesis when $\mu(X) < \infty$. This adds the implications AE $\rightarrow$ AU and AE $\rightarrow$ M. The latter implication follows transitivly from the first. The first implications was proved above as Egoroff's theorem. We obtain the following diagram as a result:



12.3. **Dominated Case.** This case assumes the hypothesis that $|f_n| \leq g$ where $g \in L^p$. In this case we can prove a version of Egoroff's theorem even though the space. Likewise we will show the implication M $\rightarrow L^p$ which will give AE $\rightarrow L^p$ and AU $\rightarrow L^p$ by transitivity. So we have the following diagram as a result:



Lemma 12.9. *Let be a space $(X, \mathcal{F}, \mu)$, and a sequence $\{f_n\}$ such that $|f_n| \leq g$ for $g \in L^p$. Then if $\{f_n\}$ converges $\mu-$a.e. then it also converges almost unifrom.*

Proof. Given a space $(X, \mathcal{F}, \mu)$, and a sequence $\{f_n\}$ such that $|f_n| \leq g$ for $g \in L^p$ and $\{f_n\}$ converges $\mu-$a.e. Let $\epsilon, \delta > 0$. Let A_1 be the set of $x \in X$ such that $\{f_n(x)\}$ does not converge to f. Then $\mu(A_1) = 0$. Now let $G_k = \{x \notin A_1 | \, |f_k(x) - f(x)| \geq \epsilon\}$. Let $E_n = \bigcup_{k \geq n} G_k$. Then $X \supset E_1 \supset E_2 \supset \ldots$. Likewise $\mu(E_1) < \infty$ as $E_1 = \{x | \, |f_k(x) - f(x)| \geq \epsilon \text{ for some } k\} \subset \{x | \, 2|g(x)| \geq \epsilon\} = \{x | \, 2^p |g(x)|^p \geq \epsilon^p\} < \infty$ since $g \in L^p$. Further $\bigcap_n E_n = \emptyset$ since $f_n \rightarrow f \mu-$a.e. Then we have that $\lim \mu(E_n) = \mu(\bigcap E_n) = 0$. Thus we can find N large enough such that $\mu(E_N) < \delta$ and $\forall n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$ by definition. $\forall k \in \mathbb{N}$ let $\epsilon_k = \frac{1}{k}$ and $\delta_k = 2^{-k}\eta$. Use the previous argument to choose sets A_k and numbers N_k such that $\mu(A_k) < \delta_k$ and $\forall n \geq N_k$ we have $|f_n(x) - f(x)| < \epsilon_k$ on A_k^c. Let $A = \bigcup A_k$. Then $\mu(A) \leq \sum 2^{-k}\eta = \eta$. Let $\epsilon > 0$. Choose k such that $\frac{1}{k} < \epsilon$, then $\forall x \in A^c \subset A_k^c$ and $n \geq N_k$ we have $|f_n(x) - f(x)| \leq \frac{1}{k} < \epsilon$. //

Lemma 12.10. *Let be a space $(X, \mathcal{F}, \mu)$, and a sequence $\{f_n\}$ such that $|f_n| \leq g$ for $g \in L^p$. Then if $\{f_n\}$ converges in measure, then it converges in L^p.*

Proof. Given a space $(X, \mathcal{F}, \mu)$, and a sequence $\{f_n\}$ such that $|f_n| \leq g$ for $g \in L^p$ and $\{f_n\}$ converges in measure. Suppose $\{f_n\}$ does not converge in mean of order p. Then there is a $\epsilon > 0$ such that for each n we can find n_k such that $\int |f_{n_k} - f|^p \geq \epsilon$. Without loss of generality we have $n_1 < n_2 < \ldots$. Then we have also that $\{f_{n_k}\}$ also converges in measure, so there is a subsequence $\{k_j\}$ such that $\{f_{n_{k_j}}\}$ converges almost uniform, whence almost everywhere to a function $f \in L^p$ since L^p is complete. Then we have that $\int f^p = lim \int f^p_{n_{k_j}}$ by the Lebesgue Dominated Convergence Theorem. Now let $g_j = |f_{n_{k_j}} - f|^p$. Then $g_j \rightarrow 0 \mu-$a.e. Let $h_j =$

$2^p(|f_{n_{k_j}}|^p + |f|^p)$ and $h = 2^{p+1}|f|^p$, then we have that $h_j \to h$ and $\int h_j \to \int h$ since $\int |f_{n_{k_j}}|^p \to \int |f|^p$. This gives then by the Lebesgue Dominated Convergence Theorem and the fact that $|g_j| \leq h_j$, that $\int |f_{n_{k_j}} - f|^p = \int g_j \to \int g = 0$. This is a contradiction.

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Note that each of the three examples above discounts a specific set of arrows. The first one removes exactly the arrows that are added in by the Finite case. The second one and the first removes the arrows that are added in by the Dominated case (because you basically pick up the Finite case from the Dominated case). The final example is never removed and this elimenates the arrows that are not in the final diagram, essentially that Reisz lemma is the best transfer of convergence upward in the diagram.

13. Product measures

Consider two complete spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$. Then we wish to define a measure space on the product $(X \times Y)$. We first wish to define our measurable sets. Naturally we would like any set of the form $A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$ to be measurable. Consider $\mathcal{R} = \{A \times B | A \in \mathcal{A}, B \in \mathcal{B}\}$. We wish to define $\lambda : \mathcal{R} \to \mathbb{R}^\infty$ such that $\lambda(A \times B) = \mu(A)\nu(B)$.

Lemma 13.1. *Let* $\{A_i \times B_i\} \subset \mathcal{R}$ *such that* $\bigcup A_i \times B_i = A \times B \in \mathcal{R}$ *Then* $\lambda(A \times B) = \sum \lambda(A_i \times B_i)$.

Consider $\mathcal{R}' = \{$ finite disjoint unions of sets in $\mathcal{R}$ $\}$. Then we have that $\mathcal{R}'$ is a algebra and we can extend define $\lambda * (E) = \int \{\sum \lambda(R_i) | R_i \in \mathcal{R}'$ and $E \subset \bigcup R_i\}$. The we have a similar 'outer measure' as we did in our construction of the lebesgue measure, and similarly to this [6] we have a measure $\mu \times \nu$ on the measureable subsets of $\mathcal{R}'$.

In order to prove the following theorems about integration on product spaces, we need the following lemma.

Theorem 13.2. *(Fubini) Given* $(X, \mathcal{A}, \mu)$ *and* $(Y, \mathcal{B}, \nu)$ *two complete spaces and* $f(x, y)$ *a* $(\mu \times \nu)-$*integrable function. Then*

1. *For almost all* x, $f_x(y) = f(x, y)$ *is* $\nu-$*integrable*
2. *For almost all* y, $f^y(x) = f(x, y)$ *is* $\mu-$*integrable*
3. $\int_Y f(x, y) d\nu$ *is a* $\mu-$*integrable function*
4. $\int_X f(x, y) d\mu$ *is a* $\nu-$*integrable function*
5. $\int_X [\int_Y f d\nu] d\mu = \int_{X \times Y} f d(\mu \times \nu) = \int_Y [\int_X f d\mu] d\nu$.

Likewise we can gain a similar theorem if we restrict our spaces and weaken the condition on f.

Theorem 13.3. *(Tonelli) Given* $(X, \mathcal{A}, \mu)$ *and* $(Y, \mathcal{B}, \nu)$ *two complete* $\sigma-$*finite spaces and* $f(x, y)$ *a* $(\mu \times \nu)-$*measurable function. Then*

1. *For almost all* x, $f_x(y) = f(x, y)$ *is* $\nu-$*measureable*
2. *For almost all* y, $f^y(x) = f(x, y)$ *is* $\mu-$*measureable*
3. $\int_Y f(x, y) d\nu$ *is a* $\mu-$*measureable function*
4. $\int_X f(x, y) d\mu$ *is a* $\nu-$*measureable function*

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[6] the proof of extensions is contained in [3], and has been omited due to time

(5) $\int_X[\int_Y f d\nu]d\mu = \int_{X\times Y} f d(\mu\times\nu) = \int_Y[\int_X f d\mu]d\nu$.

We will now use these theorems to discuss an important technique. We will define the convolution of two functions as follows:

Definition 13.4. *Given the space* $(\mathbb{R},\mathcal{M},m)$ *and two functions* f, g *such that* $f \in L^1$ *and* $g \in L^\infty \cap L^1$ *then we define for* $x \in \mathbb{R}$ *the function* $(f \star g)(x) = \int_{\mathbb{R}} f(x-y)g(x)dy$.

We note that for a given x the convolution $(f\star g)(x)$ is the area intersecting the two curves $f(t-x)$ and $g(t)$, weighted by the fuction g. We have one very nice property about convolution, and we will show how convolution can be used to solve seemingly unrelated problems.

Proposition 13.5. *On* $(\mathbb{R},\mathcal{M},m)$ *then for two functions* f, g *such that* $f \in L^1$ *and* $g \in L^\infty \cap L^1$ *convolution* $f \star g$ *is continuous*

Proof. Let $x_0 \in \mathbb{R}$. Then we will to show that for $|t| < \delta$ we have $|f\star g(x_0+t) - f\star g(x_0)| = |\int f(x_0+t-y)g(y)dy - \int f(x_0-y)g(y)dy| \le \int |g(y)||f(x_0+t-y)-f(x_0-y)|dy$. Given $\epsilon > 0$, then we can find a continuous function φ with compact support $[a,b]$ such that $\int |f-\varphi| < \frac{\epsilon}{3||g||_\infty}$. Since φ is continuous on $[a,b]$ it is uniformilly so, and we can choose δ such that when $|r-s| < \delta$ we have $|f(r)-f(s)| < \frac{\epsilon}{3(b-a)||g||_\infty}$. We would then have for $|t| < \delta$, that $|f\star g(x_0+t) - f\star g(x_0)| \le \int |g(y)||f(x_0+t-y) - f(x_0-y)|dy = \int |g(y)||f(x_0+t-y)-\varphi(x_0+t-y)+\varphi(x_0+t-y)-\varphi(x_0-y)+\varphi(x_0-y)+f(x_0-y)|dy \le \int |g(y)||f(x_0+t-y)-\varphi(x_0+t-y)| + \int_a^b |\varphi(x_0+t-y)-\varphi(x_0-y)|dy + \int |\varphi(x_0-y)+f(x_0-y)|dy \le ||g||_\infty \frac{\epsilon}{3||g||_\infty} + ||g||_\infty \int_a^b \frac{\epsilon}{(b-a)||g||_\infty}dy + ||g||_\infty \frac{\epsilon}{3||g||_\infty} < \epsilon$. //

We can use this to show the following:

Proposition 13.6. *Given sets* E, F *such that* $m(E) > 0$ *and* $m(F) > 0$ *then we have that* $E+F$ *contains an interval.*

Proof. First consider $\chi_E \star \chi_F$. If either $m(E) = \infty$ or $m(F) = \infty$ then choose finite subsets of them, so without loss of generality $m(E) < \infty$ and $m(F) < \infty$. This gives that both χ_E and χ_F are integrable and essentially bounded. We have that $\int_X \chi_E\star\chi_F dx = \int_X\int_Y \chi_E(x-y)\chi_F(y)dydx = \int_Y\int_X \chi_E(x-y)\chi_F(y)dxdy$ by Fubini. We have also that $\int_X \chi_E\star\chi_F dx = \int_Y\int_X \chi_E(x-y)\chi_F(y)dxdy = \int_Y \chi_F(y)\int_X \chi_E(x-y)dxdy = \int_Y \chi_F(y)m(E)dy = m(E)m(F) > 0$, so since $\int_X \chi_E\star\chi_F dx > 0$ we have that $\chi_E\star\chi_F > 0$ at some point, say x_0. Since convolution is continuous we must have that $\chi_E\star\chi_F > 0$ on some interval (a,b). For any $t \in (a,b)$ since $\chi_E\star\chi_F(t) = \int \chi_E(t-y)\chi_F(y)dy > 0$ we have that $\chi_E(t-y)\chi_F(y) > 0$ for some y. This condition gives that $y \in F$ and $t-y \in E$ which says that $t = t-y+y \in E+F$, hence $(a,b) \subset E+F$. //

14. Assignments

Homework assignments were composed of problems that were to be graded and problems that were not required. I will list the problems (almost all out of [3], the complement being a set of measure zero), by bolding the problems that were assigned.

Assignment 0

Let $f = g \circ h$ where h is a Riemann integrable function on $[a, b]$, $h([a, b]) \subset [s, t]$, and g is continuous on $[s, t]$. Then f is Riemann integrable.

Proof. Since $[s, t]$ is compact we know that g attains its maximum value on the interval $[s, t]$, so let $M = sup_{x\in[s,t]}|g(x)|$. Let $\epsilon > 0$. We seek a partition P of $[a, b]$ so that $U_f(P) - L_f(P) \leq \epsilon$. Since g is continuous on a compact set, it is uniformly so, thus we can find a δ so that for any $y, z \in [s, t]$ such that if $|y - z| \leq \delta$, then $|g(y) - g(z)| \leq \frac{\epsilon}{2(b-a)}$. Now since h is Riemann integrable, we can find a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ so that $\sum_{\{k|M_k(h)-m_k(h)>\delta\}} \leq \frac{\epsilon}{4M}$. Then we consider the difference $U_f(P) - L_f(P) = \sum_{k=1}^{n} M_k(f) - m_k(f)\triangle x_k$. We split this sum into two categories. Let $k \in G$ if $M_k(h) - m_k(h) \leq \delta$ and $k \in B$ otherwise. For $\sum_{k\in G} M_k(f) - m_k(f)\triangle x_k$ we have that $M_k(h) - m_k(h) \leq \delta \Rightarrow M_k(f) - m_k(f) \leq \frac{\epsilon}{2(b-a)}$. Thus $\sum_{k\in G} M_k(f) - m_k(f)\triangle x_k \leq \frac{\epsilon}{2(b-a)} \sum_{k\in G} \triangle x_k \leq \frac{\epsilon}{2(b-a)} * (b-a) = \frac{\epsilon}{2}$. For $\sum_{k\in B} M_k(f) - m_k(f)\triangle x_k$, we know that at worst $M_k(f) - m_k(f) \leq 2M$, so we have that $\sum_{k\in B} M_k(f) - m_k(f)\triangle x_k \leq 2M \sum_{k\in B} \triangle x_k$, but by our choice of partition $\sum_{k\in B} \triangle x_k \leq \frac{\epsilon}{4M}$ so we have $\sum_{k\in B} M_k(f) - m_k(f)\triangle x_k \leq 2M * \frac{\epsilon}{4M} = \frac{\epsilon}{2}$. Thus $U_f(P) - L_f(P) = \sum_{k=1}^{n} M_k(f) - m_k(f)\triangle x_k = \sum_{k\in G} M_k - m_k\triangle x_k + \sum_{k\in B} M_k - m_k\triangle x_k \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ which shows that f is Riemann integrable over $[a, b]$. //

Assignment 1

Problems **19/20**, **58/5**, 58/6

1) Let C be a nonempty family of subsets of X, and let $\mathfrak{A}$ bethe algebra of subsets of X generated by C. Let $\varepsilon_0 = C$. Let $C^* = \{X \setminus A | A \in C\}$. Then define $\varepsilon_1 = C \cup C^*$. Let $\varepsilon_2 = \{A_1 \cap \ldots \cap A_n | A_j \in \varepsilon_1\}$, and $\varepsilon_3 = \{A_1 \cup \ldots \cup A_m | A_k \in \varepsilon_2\}$. We have that $C = \varepsilon_0 \subset \varepsilon_1 \subset \varepsilon_2 \subset \varepsilon_3 \subset \mathfrak{A}$. We now show that $\varepsilon_3 = \mathfrak{A}$.

Proof. We show that $\mathfrak{A} \subset \varepsilon_3$ by showing that ε_3 is an algebra containing C and is contained in any other algebra containing C.

However first we show that ε_3 is an algebra. Let $A_1 \cup \ldots \cup A_n$ and $B_1 \cup \ldots \cup B_m$ both be elements of ε_3, then $A_1 \cup \ldots \cup A_n \cup B_1 \cup \ldots \cup B_m \in \varepsilon_3$ as each A_i and B_j are in ε_2, thus ε_3 is closed under union. Similarly we can show that ε_3 is closed under intersection. Let $A_1 \cup \ldots \cup A_n$ and $B_1 \cup \ldots \cup B_m \in \varepsilon_3$, then $(A_1 \cup \ldots \cup A_n) \cap (B_1 \cup \ldots \cup B_m) = (A_1 \cup \ldots \cup A_n) \cap B_1) \cup \ldots \cup (A_1 \cup \ldots \cup A_n) \cap B_m) = (A_1 \cap B_1) \cup \ldots \cup (A_n \cap B_1) \cup \ldots \cup (A_1 \cap B_m) \cup \ldots \cup (A_n \cap B_m) \in \varepsilon_3$ as each $A_i \cap B_j \in \varepsilon_2 \forall i, j$ Now let $A_1 \cup \ldots \cup A_n \in \varepsilon_3$, then $(A_1 \cup \ldots \cup A_n)^c = A_1^c \cap \ldots \cap A_n^c$. However since each $A_i \in \varepsilon_2 \Rightarrow A_i = B_{i,1} \cap \ldots \cap B_{i,m_i}$, therefore $A_1^c \cap \ldots \cap A_n^c = (B_{1,1}^c \cup \ldots \cup B_{1,m_1}^c \cap B_{2,1}^c \cup \ldots \cup B_{2,m_2} \cap B_{3,1} \cup \ldots \cup B_{n-1,m_{n-1}}^c \cap B_{n,1}^c \cup \ldots \cup B_{n,m_n}^c)$. With $B_{i,j} \in \varepsilon_1 \subset \varepsilon_3$. Since we have already shown ε_3 to be closed under both union and intersection, we have now that ε_3 is closed under complement. and is so an algebra. Now let $\mathfrak{A}'$ be any algebra such that $C \subset \mathfrak{A}'$. Then since $C \subset \mathfrak{A}' \Rightarrow C^* \subset \mathfrak{A}'$. This gives then that $A_1 \cap \ldots \cap A_n \in \mathfrak{A}'$ where $A_i \in \varepsilon_1$ which gives that $\varepsilon_2 \subset \mathfrak{A}'$. Again since $\mathfrak{A}$ is an algebra, we have that $A_1 \cup \ldots \cup A_m \in \mathfrak{A}'$ for $A_i \in \varepsilon_2$ which gives that $\varepsilon_3 \subset \mathfrak{A}'$ which completes the proof. //

19/20) Let C be a collection of sets and denote the smallest $\sigma-$algebracontaining C to be $\sigma(C)$. Let $E \in \sigma(C)$. Then we show that there is a $\sigma-$algebra generated by a countable collection $C' \subset C$ such that $E \in \sigma(C')$.

Proof. Consider $\sigma_c(C) = \bigcup_{C'} \sigma(C')$ where C' ranges over all countable subsets of C. If we can show that $\sigma(C) \subset \sigma_c(C)$ then $\forall E \in \sigma(C)$ we would have $E \in \sigma_c(C)$ which gives that there is a countable collection $C' \subset C$ such that $E \in \sigma(C')$. We show that $\sigma(C) \subset \sigma_c(C)$ by showing that $\sigma_c(C)$ is a $\sigma-$algebra containing C. First $C \subset \sigma_c(C)$ as $\forall c \in C$ then $\sigma(\{c\}) \subset \sigma_c(C)$ which gives $c \in \sigma_c(C)$. Now, let $A \in \sigma_c(C)$. Then $A \in \sigma(C')$ for some countable $C' \subset C$, then $A^c \in \sigma(C') \Rightarrow A^c \in \sigma_c(C)$. Let $\{A_i\}_{i=1}^{\infty} \subset \sigma_c(C)$ be a countable collection of sets. Then each $A_i \in \sigma(C_i)$ for some countable collection $C_i \subset C$. Let $C' = \bigcup_{i=1}^{\infty} C_i$. Then $C' \subset C$ is countable, and $A_i \in \sigma(C') \forall i$. This gives that $\bigcup_{i=1}^{\infty} A_i \in \sigma(C') \subset \sigma_c(C)$. This gives that $\sigma_c(C)$ is a $\sigma-$algebra containing C, thus $\sigma(C) \subset \sigma_c(C)$.

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58/5) Let $A = \mathbb{Q} \cap [0, 1]$. Let $\{I_n\}_{n=1}^k$ be a finite collection of open intervals covering A. Then $\sum l(I_n) \geq 1$.

Proof. Let $\{I_n = (a_n, b_n)\}_{n=1}^k$ be a finite collection of open intervalscovering A. Then some element $(a_j, b_j) \in \{I_n\}_{n=1}^k$ must be such that $a_j \leq 0$. As otherwise $a_n > 0 \forall n$. Let $m = min\{a_1, \ldots, a_k\}$. Then since $\mathbb{Q}$ is dense, any $q \in (0, a_m) \cap \mathbb{Q}$ would not be covered. So up to renumbering let $a_1 \leq 0$. Now we can repeat the same argument for b_1. If b_1 is rational, then it must be contained in one of the intervals, and without loss of generality we would have $a_2 < b_1$. If b_1 is irrational, then a similar argument shows that at least one of the $a_n' s$ must be such that $a_n \leq b_1$. Either way let $I_2 = (a_2, b_2)$ be such that $a_2 \leq b_1$. We can continue one, and since our collection of intervals is finite we must terminate. More importantly, without loss of generality we must have $a_k \geq 1$. This is because if $a_m = max\{a_1, \ldots, a_k\}$, and $a_m < 1$, then we can find $q \in (a_m, 1)$ that is not covered. So we have finally that $\{I_n\}_{n=1}^{\infty}$ such that $a_1 \leq 0$ and $b_k \geq 1$, and $a_i \leq b_i - 1$. This gives that $\sum l(I_n) = b_1 - a_1 \ldots b_k - a_k = b_k - a_1 + b_1 + b_2 - a_2 + \ldots + a_{k-1} - a_k \geq b_k - a_1 \geq 1$. //

4) Let $\{E_n\}$ be an infinite sequence of measurable sets. We define $\overline{lim} E_n = \{x | x \in E_n \text{ for infinitely many } n\}$ and $\underline{lim} E_n = \{x | x \in E_n \text{ for all but finitely many } n\}$.

a) We will give $\overline{lim} E_n$ and $\underline{lim} E_n$ in terms of unions and intersections of elements of $\{E_n\}$. We recall the definitions for $\overline{lim}$ and $\underline{lim}$ for sequences of points x_n. Namely that

$$\overline{lim} x_n = \inf_{n>0} \sup_{k \geq n} (x_k) \text{ and } \underline{lim} x_n = \sup_{n>0} \inf_{k \geq n} x_k$$

If we extend these for our sequence of sets we see that we should have:

$$\overline{lim} E_n = \inf_{n>0} \sup_{k \geq n} (E_k) \text{ and } \underline{lim} E_n = \sup_{n>0} \inf_{k \geq n} E_k$$

Now we recall that for a collection of sets $\{E_n\}$, then $\sup\{E_n\} = \bigcup_n E_n$ and $\inf\{E_n\} = \bigcap_n E_n$. So we make the following claim:

$$\overline{lim} E_n = \bigcap_{n>0} \bigcup_{k \geq n} E_k \text{ and } \underline{lim} E_n = \bigcup_{n>0} \bigcap_{k \geq n} E_k$$

Proof. Let $x \in \overline{lim} E_n$. Then $\forall n > 0, \exists k \leq n$ such that $x \in E_k \Rightarrow x \in \bigcup_{k \leq n} E_k \Rightarrow x \in \bigcap_{n>0} \bigcup_{k \leq n} E_k$. Conversly let $x \in \bigcap_{n>0} \bigcup_{k \leq n} E_k$, then $\forall n > 0, x \in \bigcup_{k \leq n} E_k$ which gives that there is a $k \leq n$ such that $x \in E_k$ or that $x \in \overline{lim} E_n$. Dually $\underline{lim} E_n = \bigcup_{n>0} \bigcap_{k \leq n} E_k$ as $x \in \underline{lim} E_n$ gives that $x \notin E_n$ for finitely many n. Let m be the maximal index so that $x \notin E_m$. Then $\forall k \geq m x \in E_k \Rightarrow \bigcap_{k \geq m} E_k \Rightarrow x \in \bigcup_{n>0} \bigcap_{k \geq n} E_k$. Also if $x \in \bigcup_{n>0} \bigcap_{k \geq n} E_k \Rightarrow \exists n$ such that $\forall k \geq n, x \in \bigcap_{k \geq n} E_k$ This gives that the collection of indicies for which $x \notin E_k$ is limited to $\{1, 2, \ldots, n\}$, which is thereby finite, so $x \in \underline{lim} E_n$.

//

b) We will show that $\overline{lim}\ m(E_n) \leq m(\overline{lim} E_n)$ given that $E_n \subset A$ and $m(A)$ is finite.

Proof. Now using the definitions in part (a), we can show that $\overline{lim}\ m(E_n) \leq m(\overline{lim} E_n)$. We note that $\bigcup_{k \geq n+1} E_k \subset \bigcup_{k \geq n} E_k$ so that $\{\bigcup_{k \geq n} E_k\}_n$ is a decreasing sequence. Now we start with $m(\overline{lim} E_n) = m(\bigcap_{n>0} \bigcup_{k \geq n} E_k) = \lim_{n \to \infty} m(\bigcup_{k \geq n} E_k)$ by proposition. Further by subaddativity we have $\lim_{n \to \infty} m(\bigcup_{k \geq n} E_k) \geq \lim_{n \to \infty} \sum_{k \geq n} m(E_k) \geq \lim_{n \to \infty} \sup_{k \geq n} m(E_k) = \overline{lim}\ m(E_n)$. //

Assignment 2

Problems 71/22, 71/23, 71/24, 71/25, **71/28** (assume 46/2.37, 50/2.48, 64/3.14(a), 66/3.16), **85/2(b)**, **89/9**

Exercise 71/28 Let C be the Cantor Set. Now for each $x \in [0,1]$ write x with its ternary expansion $x = a_1 a_2 \ldots$. Define $N = \infty$ if $a_n \neq 1 \forall n$. If this condition is not met, let N be minimal such that $a_N = 1$. Define $b_n = \frac{1}{2}a_n$ for $n < N$ and $b_N = 1$. Define $f_C(x) = \sum_{i=1}^{N} \frac{b^n}{2^n}$. Then f_C is continuous, monotone, and constant on each interval $I \subset [0,1]\ C$. Define $f\colon [0,1] \to [0,2]$ to be $f(x) = f_C(x) + x$, Then:

a) f is a homeomorphism

Proof. We show that f is a bijective continuous and open. f is continuous as it is the sum of two continuous functions. f is injective as $x \neq y$ then without loss of generality $x < y$, and we have that $f_C(x) \leq f_C(y) \Rightarrow f_C(x)+x < f_C(y)+y \Rightarrow f(x) \neq f(y)$. f is surjective as $f(0) = f_C(0)+0 = 0$, $f(1) = f_C(1)+1$ but $f_C(1) = \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} - 1 = \frac{1}{1-\frac{1}{2}} - 1 = 2-1 = 1$, thus $f(1) = 1+1 = 2$, and so since f is continuous by the intermediate value property, f is surjective. We now show that f is open. Any open subset of $[0,1]$ is generated by the basis elements $[0,a), (b,1], (a,b)$ for $a, b \in [0,1]$. Now $f([0,a)) = [0, f(a)), f((b,1]) = (f(b), 2], f(a,b) = (f(a), f(b))$ since f is continuous and monotone, again using the intermediate value property. Thus f is an open map, and is so a homeomorphism. //

b) $m(f(C)) = 1$

Proof. We know that $m(C) = 0$ which gives that $m([0,1] \setminus C) = 1$. Further since f is a bijection we know that $f(C) \cup [0,2] \setminus f(C) = [0,2]$, so $m(f(C)) + m([0,2] \setminus f(C)) = m([0,2]) = 2$. Now we can write $[0,1] \setminus C = \bigcup_i (a_i, b_i)$ as a disjoint union of open intervals. This gives that $[0,2] \setminus f(C) = f([0,1] \setminus C) = f(\bigcup_i (a_i, b_i)) = \bigcup_i f(a_i, b_i)$ as f is a bijection. Now since $x \in (a_i, b_i) \Rightarrow f_C(x) = c_i$, then we have that $[0,2] \setminus f(C) = \bigcup_i f(a_i, b_i) = \bigcup_i (a_i + c_i, b_i + c_i)$. However this gives that $m([0,2] \setminus f(C)) = m(\bigcup_i (a_i + c_i, b_i + c_i)) = \sum_i m(a_i + c_i, b_i + c_i) = \sum_i m(a_i, b_i) = m([0,1] \setminus C) = 1$. Thus we have that $m(f(C)) = 2 - m([0,2] \setminus f(C)) = 2 - 1 = 1$. //

c) Let $g = f^{-1}$, then there is a measurable set A such that $g^{-1}(A')$ is not measurable

Proof. Consider $f(C)$ in the previous problem. Then $m(f(C)) = 1$, so there is a non-measureable subset, say $A' \subset f(C)$. Let $A = g(A')$. Then since f is a bijection, so is g, hence $A \subset C$ which gives that A is measurable. However $g^{-1}(A) = A'$ which by construction is not measureable. //

d) There is a measurable function h such that $h \circ g$ is not measureable

Proof. Let $h = \chi_A$. Then $h : [0,1] \to \mathbb{R}$ is measurable and we have that $h \circ g$ is not measurable. This is so as $(h \circ g)^{-1}((\frac{1}{2}, \infty)) = g^{-1}(h^{-1}(\frac{1}{2}, \infty)) = g^{-1}(A) = A'$ which is not measurable. //

e) There is a measurable set which is not Borel

Proof. We know that A is measurable, however A cannot be a Borel set. We begin by noting that the continuous inverse image of a Borel set is Borel, as continuous inverse images preserve set operations and open sets. Therefore since g is continuous, we have that $g^{-1}(A) = A'$ would have to be Borel if A were, however A' is not measurable and in so cannot be Borel. //

Exercise 85/2b: (Lebesgue's Theorem) A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable iff f is continuous almost everywhere

Proof. ($\Rightarrow$)Let f be Riemann integrable. Let $D = \{x|f \text{ is not continuous at } x\}$. Then we show that $\forall \eta > 0$, then $m(D) < \eta$. This gives that f is continuous a.e. Let $\eta > 0$, and choose $\{\lambda_j\}, \{\mu_j\}$ positive real sequences such that $\{\lambda_j\} \to 0$ and $\sum_j \mu_j < \eta$. Then $\forall j$ we can find a partition P_j such that $\sum_{\{k|M_{k,j} - m_{k,j} > \lambda_j\}} \triangle x_{k,j} < \mu_j$. Let $U_j = \bigcup_{\{k|M_{k,j} - m_{k,j} \le \lambda_j\}} (x_{k-1,j}, x_{k,j})$. Now let $G = \bigcap_{j=0}^{\infty} U_j$. Then $\forall x \in G, x \in U_j \forall j$. Then $\forall \epsilon > 0$, choose j such that $\lambda_j < \epsilon$. then we have that $x \in U_j \Rightarrow \exists k$ such that $x \in (x_{k-1,j}, x_{k,j})$ such that $\forall y \in (x_{k-1,j}, x_{k,j})$ we have $|f(x) - f(y)| \le M_{k,j} - m_{k,j} \le \lambda_j < \epsilon$, so f is continuous at x. This gives that $D \subset [a, b] \setminus G$. Now $m([a, b] \setminus G) = m([a, b] \setminus \bigcap_{j=0}^{\infty} U_j) = m(\bigcup_{j=0}^{\infty} [a, b] \setminus U_j) \le \sum_{j=0}^{\infty} m([a, b] \setminus U_j) \le \sum_{j=0}^{\infty} \mu_j < \eta$. Thus f is continuous a.e

($\Leftarrow$)Now assume f to be continuous a.e., and let $\lambda, \mu > 0$ be given. Then let $D = \{x|f \text{ is not continuous at } x\}$ Let U be an open set such that $D \subset U$ and $m(U) < \mu$. Now consider $F = [a, b] \setminus U$. Then F is both closed and bounded, hence compact. Now for each point $p \in F$, choose δ_p such that $|x - p| \le \delta_p$, then $|f(x) - f(p)| < \frac{\lambda}{2}$, which we can do because f is continuous on F. Then this forms an open cover, to which there must be a finite subcover, generated by $p_1, \ldots, p_n$ in F. Let $P = \{x_i\}_{i=0}^{n}$ be the partition generated by the endpoints $(p_j - \delta_{p_j}, p_j + \delta_{p_j}) \cap [a, b]$. Then P is a partition of $[a, b]$. Now given $(x_{k-1}, x_k) \subset F$, then $M_k - m_k \le \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda$, thus $\bigcup_{\{k|M_k - m_k > \lambda\}} (x_{k-1}, x_k) \subset U$, and since $m(U) < \mu$ we have that f is Riemann integrable.

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Exercise 89/9: Let $\{f_n : (-\infty, \infty) \to \mathbb{R}\}$ be a sequence of non-negative measurable functions such that $f_n \to f$ a.e. and $\int f_n \to \int f < \infty$. Then we show that for each measurable set E, we have that $\int_E (f_n) \to \int_E f$.

Proof. Let E be a measurable set. By Fatou's lemma we have $\int_E f \le \underline{\lim} \int_E f_n$. Now we consider $\overline{\lim} \int_E f_n = \overline{\lim}(\int f_n - \int_{E^c} f_n) \le \overline{\lim}(\int f_n) + \overline{\lim}(-\int_{E^c} f_n) = \int f - \underline{\lim}(\int_{E^c} f_n)$. Again Fatou gives us $\int_{E^c} f \le \underline{\lim}(\int_{E^c} f_n) \Rightarrow -\int_{E^c} f \ge -\underline{\lim}(\int_{E^c} f_n)$, thus we have $\int f - \underline{\lim}(\int_{E^c} f_n) \le \int f - \int_{E^c} f = \int_E f$. Thus $\overline{\lim} \int_E f_n \le \int_E f \Rightarrow \int_E f_n \to \int_E f$. //

Assignment 3

Problems 89/4, 93/15, **94/16**, 94/18, 94/19, **102/4**, 104/5, **105/10**

Exercise 94/18 We prove the Riemann-Lebesgue theorem. Let f be an integrable function on $\mathbb{R}$. Then $lim_{n\to\infty}\int f(x)*cos(nx)=0$

Proof. Let $\epsilon>0$. Choose a step function $\phi=\sum_{i=1}^{k}c_i\chi_{[a_i,b_i]}$ such that $\int|f-\phi|<\frac{\epsilon}{2}$. Then we have that $\int f-\int\phi<\frac{\epsilon}{2}$, and so $\int f*cos(nx)-\int\phi*cos(nx)<\frac{\epsilon}{2}$. Then we can integrate $\int\phi*cos(nx)=\frac{1}{n}\sum_{i=1}^{k}(sin(nb_i)-sin(na_i)$. Choose N such that $\forall n\geq N$ we have $\int\phi*cos(nx)=\frac{1}{n}\sum_{i=1}^{k}(sin(nb_i)-sin(na_i)\leq\frac{2k}{n}<\frac{\epsilon}{2}$, then $\forall n\geq N$, $\int f*cos(nx)\leq\int\phi*cos(nx)+\frac{\epsilon}{2}\leq\epsilon$ so $\lim_{n\to\infty}\int f*cos(nx)=0$.

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Exercise 102/4: Let f be such that f is continuous on $[a.b]$ and $D^+f\geq 0$, then f is nondecreasing.

Proof. It suffices to assume that $D^+f\geq\epsilon$ for some $\epsilon>0$ as given f in the description then f is nondecreasing exactly when $f+\epsilon*x$ is nondecreasing and $D^+(f+\epsilon*x)=\overline{\lim_{h\to 0^+}}\frac{f(x+h)+\epsilon*(x+h)-f(x)-\epsilon*h}{h}=\overline{\lim_{h\to 0^+}}\frac{f(x+h)-f(x)}{h}+\epsilon\geq\epsilon$. So let f be such that $D^+f\geq\epsilon$. Let $x\leq y$. Then $[x,y]\subset[a,b]$. We show that f must attain its maximum on $[x,y]$ at y. Suppose f attains its maximum at $x'\in[x,y)$. Then $f(x'+h)\leq f(x')$ which gives that $f(x'+h)-f(x')\leq 0\Rightarrow\overline{\lim_{h\to 0^+}}\frac{f(x'+h)-f(x')}{h}\leq 0$ which is a contradiction. //

Exercise 104/10:

(1) $f(x)=\begin{cases}x^2*sin(\frac{1}{x^2}) & \text{if } x\neq 0\\ 0 & x=0\end{cases}$ is not of bounded variation on $[-1,1]$

Proof. Consider the points $x_i=\sqrt{\frac{2}{\pi(2i+1)}}$. Then we compute $T_{-1}^{1}(f,P)$ where P is the partition consisting of the points x_i. We consider $\sum_{i=1}^{\infty}|x_i^2sin(\frac{1}{x_i^2})-x_{i-1}^2sin(\frac{1}{x_{i-1}^2})|=\frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{2i+1}+\frac{1}{2i-1}=\frac{2}{\pi}(\sum_{i=1}^{\infty}\frac{1}{2i+1}+\sum_{i=1}^{\infty}\frac{1}{2i-1})=\frac{2}{\pi}(\sum_{i=1}^{\infty}\frac{1}{2i+1}+\sum_{i=0}^{\infty}\frac{1}{2i+1})=\frac{2}{\pi}(1+2\sum_{i=1}^{\infty}\frac{1}{2i+1})\geq\frac{2}{\pi}(1+\sum_{i=1}^{\infty}\frac{1}{i+1})$, as $\frac{2}{2i+1}\geq\frac{1}{i+1}$, However $\frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{i+1}$ diverges by comparision to a harmonic series, thus our variation for this partition is unbounded and our function is not of bounded variation. //

(2)

$$f(x)=\begin{cases}x^2*sin(\frac{1}{x}) & \text{if } x\neq 0\\ 0 & x=0\end{cases}$$

is of bounded variation on $[-1,1]$

Proof. So our function is continuous on $[-1,1]$ and differentiable atany $x\neq 0$, and since $lim_{t\to 0}\frac{t^2sin(\frac{1}{t})}{t}=tsin(\frac{1}{t})\to 0$ we have that f is differentiable at $x=0$. Further f' is Riemann integrable since it is discontinuous at only one point, and since we have $|x*cos(\frac{1}{x^2})*(\frac{-2}{x})+sin(\frac{1}{x^2})|=|sin(\frac{1}{x^2})-2*cos(\frac{1}{x^2})|\leq 3$ we have that f' is bounded and so f is of bounded variation.

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Assignment 4

Problems 110/12, 110/14, 111/16, 111/17, **112/20**, **116/26**, 117/27, 119/1, **119/2**, 119/3, 119/4, 126/13, **127/17**

Exercise 112/20

(1) Every lipschitz function is absolutly continuous

Proof. Let f be such that $|f(x) - f(y)| \leq M|x-y|$. Now $\epsilon > 0$ and choose $\delta < \frac{\epsilon}{M}$. Then we have that for $\sum_{i=1}^{n} |x_i' - x_i|$ then $\sum_{i=1}^{n} |f(x_i') - f(x_i)| \leq \sum_{i=1}^{n} M|x_i' - x_i| \leq M\delta = \epsilon$. So f is absolutly continuous. //

(2) For absolutly continuous functions lipschitz is equivalent to having a bounded derivative

Proof. Any lipschitz function f has a bounded derivative as $|f(x)-f(y)| \leq M|x-y| \Rightarrow \frac{|f(x)-f(y)}{|x-y|} \leq M$ for any x, y. So it suffices to show that an absolutly continuous function with bounded derivative is lipschitz. So we must show that $|f(x)-f(y)| \leq M|x-y|$. Without loss of generality choose $x = a$ and $y = b$ as for any smaller interval all of the same properties hold. Let $\epsilon > 0$ and let δ be the corresponding tolerence for f to be absolutely continuous. Let $l([a,b]) = \{x | \forall y \in (x, x+h)$ we have $\frac{f(x)-f(y)}{x-y} \leq M\}$. This is a vitali covering, so we can choose $\{(x_i, y_i)\}_{i=0}^{n} + 1$ such $a = y_0 \leq x_1 < y_1 \leq x_2 < \ldots \leq x_{n+1} = b$ so that $\sum_{k=0}^{n} |x_{k+1} - y_k| < \delta$. So now we have that $\sum_{k=0}^{n} |f(x_{k+1}) - f(y_k)| < \epsilon$, and $|f(x_i) - f(y_i)| \leq M|x_i - y_i$. Thus we have $|f(x) - f(y)| = |sum_{k=0}^{n} |f(x_{k+1}) - f(y_k)| + \sum_{k=1}^{n} |f(x_k) - f(y_k)| \leq \epsilon + M\sum_{k=1}^{n} |x_k - y_k| \leq \epsilon + M|x-y|$. And since $epsilon$ was arbitrary $|f(x) - f(y)| \leq M|x-y|$ which gives that f is lipschitz. //

(3) A function with D^+ bounded is lipschitz

Proof. We have that if f has a bounded derivate then, f must be of bounded variation since we can choose a partition $\pi = \{a = x_0, x_1, \ldots, x_n = b\}$ so that $|f(x_i) - f(x_{i+1}| \leq M$. Therefore f is given as a difference of nondecreasing functions. So it suffices to assume that f is nondecreasing. Given this, for each n choose $x_n > y_n$ such that $\frac{f(x_n)-f(y_n)}{x_n - y_n} > n$. Without loss of generality let $(x_n, y_n) \to (x^*, y^*)$. Then since $|f(x)| < K$ we have that $|f(x_n) - f(y_n)| < 2K$ so because $\frac{f(x_n)-f(y_n)}{x_n-y_n} > n$ we have that $x_n - y_n \to 0$ or that $x^* = y^*$. However this contradicts the fact that D^+ must be bounded at x^* so we have a contradiction, and thus f must be lipschitz. //

Exercise 116/26 For which functions f do we have $\int e^f = e^{\int f}$ on $[0,1]$

Proof. We note that any function which is constant almost everywhere provides equality. Now let f be given and let $\alpha = \int f$. Then we choose m so that $e^f \geq m(f-\alpha) + e^{\alpha}$. Now let $E = \{f \neq \alpha\}$. Then if f is not constant almost everywhere, we have that $m(E) > 0$. then we know that $e^f - m(f-\alpha) - e^{\alpha} > 0$ on E. Integrating gives a strictly positive result, which gives strict inequality above. //

Exercise 119/2 Show that for f bounded measurable we have $lim_{p\to\infty} ||f||_p = ||f||_\infty$

Proof. First note that since $|f| \leq ||f||_\infty$ we have that $||f||_p = (\int |f|^p)^{\frac{1}{p}} \leq (\int ||f||_\infty^p)^{\frac{1}{p}} = ||f||_\infty$ so our sequence is bounded. Likewise for $p_1 < p_2$, let $p = \frac{p_2}{p_1} > 1$. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we have that $||f||_{p_1}^{p_1} = \int |f|^{p_1} = \int |f|^{p_1} * 1 \leq ||f^{p_1}||_p * ||1||_q = ||f^{p_1}||_p$ by Hölder's inequality. Now $||f^{p_1}||_p = (\int (|f|_1^p)^p)^{\frac{1}{p}} = ((\int |f|_2^p)^{\frac{1}{p_2}})_1^p = ||f||_{p_2}^{p_1}$ which gives that $||f||_{p_1} \leq ||f||_{p_2}$ so our sequence is increasing, and hence convergent. This gives that the sequence $\{\frac{||f||_p}{||f||_\infty}\}$ is also convergent and that its limit is less than 1. If we have that the limit is 1 then we are done. So we will show that the limit is greater than $1 - \epsilon$ for any ϵ. Let $\epsilon > 0$. Then there exists E a set such that $m(E) > 0$ and $\frac{|f|}{||f||_\infty} > 1 - \epsilon$. This gives that $(\int (\frac{|f|}{||f||_\infty})^p)^{\frac{1}{p}} \geq (\int_E (\frac{|f|}{||f||_\infty)^p})^{\frac{1}{p}} \geq (1-\epsilon)(m(E))^{\frac{1}{p}}$ which as $p \to \infty$ then $(1-\epsilon)(m(E))^{\frac{1}{p}} \to 1 - \epsilon$ thus $lim_{p\to\infty} \frac{||f||_p}{||f||_\infty} > 1 - \epsilon$. This gives the result. //

Exercise 127/17 Let $f_n \subset L^p$ be such that $f_n \to f$ a.e. Then for $g \in L^q$ where $\frac{1}{p} + \frac{1}{q} = 1$, we have that $lim \int f_n g = \int fg$

Proof. Let $\{f_n\} \subset L^p$, $||f_n||_p \leq M||$, and $f_n \to f \in L^p$ a.e. Let $g \in L^q$. Then since $m([0,1]) < \infty$ we have that $g \in L^1$ and so $\int |g| = \alpha < \infty$. Likewise $g \in L^q \Rightarrow |g|^q \in L^1$ so we can find a number $\eta > 0$ such that if A is a set with $m(A) < \eta$ then $\int_A |g|^q < \frac{\epsilon^q}{2^q(M+||f||_p)^q}$. Now since we are again in a finite space, Egoroff gives that we can find a set A such that $m(A) < \eta$ and $f_n \to f$ uniformly on A^c. Then choose N such that $n \geq N$ gives $|f_n - f| \leq \frac{\epsilon}{2(\alpha)m(A^c)}$ and we have that $\int |f_n g - fg| = \int |f_n - f||g| = \int_A |f_n - f||g| + \int_{A^c} |f_n - f||g| \leq (\int_A |f_n - f|^p)^{\frac{1}{p}} (\int_A |g|^q)^{\frac{1}{q}} + \frac{\epsilon}{2\alpha m(A^c)} \int_{A^c} |g| \leq (||f_n||_p + ||f||_p)(\frac{\epsilon^q}{2^q(M+||f||_p)^q})^{\frac{1}{q}} + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. //

Assignment 5

Problems: **130/19**, **134/21**, 135/22, 135/23, 135/24, 258/5, **263/14**, 268/21, 268/22, **270/24**, **270/25**, **270/26**.

Exercise 130/19

For a partition $\triangle = (\xi_0, \ldots, \xi_n)$ let $T_\triangle(f)$ denote the $\triangle$-approximate. Show $||T_\triangle(f)||_p \leq ||f||_p$ for $1 \leq p < \infty$.

Proof. Let $\triangle\xi_k = \xi_{k+1} - \xi_k$, and $\alpha_k = \int_{\xi_k}^{\xi_{k+1}} f$. Our proof is split into two cases. Case $p = 1$: Consier $||T_\triangle(f)||_1 = \sum_{k=0}^{n-1} \int_{\xi_k}^{\xi_{k+1}} |\frac{\alpha_k}{\triangle\xi_k}| = \sum_{k=0}^{n-1} |\int_{\xi_k}^{\xi_{k+1}} f| \leq \sum_{k=0}^{n-1} \int_{\xi_k}^{\xi_{k+1}} |f| = ||f||_1$. Case $p > 1$: Consider $||T_\triangle(f)||_p^p = \int |T_\triangle(f) = \sum_{k=0}^{n-1} \int_{\xi_k}^{\xi_{k+1}} |T_\triangle(f)|^p = \sum_{k=0}^{n-1} \frac{|\alpha_k|^p}{(\triangle\xi_k)^{p-1}} = \sum_{k=0}^{n-1} \frac{|\int_{\xi_k}^{\xi_{k+1}} f^p|}{(\triangle\xi_k)^{p-1}} \leq \sum_{k=0}^{n-1} \frac{\int_{\xi_k}^{\xi_{k+1}} |f|^p}{(\triangle\xi_k)^{p-1}}$ Now let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we have that by Hölder's inequality $\int_{\xi_k}^{\xi_{k+1}} |f| \leq ||f||_p * ||q||_q$, so we have that $\sum_{k=0}^{n-1} \frac{((\int_{\xi_k}^{\xi_{k+1}} |f|^p)^{\frac{1}{p}} (\int_{\xi_k}^{\xi_{k+1}} 1^q)^{\frac{1}{q}})^p}{(\triangle\xi_k)^{p-1}} = \sum_{k=0}^{n-1} \frac{((\int_{\xi_k}^{\xi_{k+1}} |f|^p)^{\frac{1}{p}} (\triangle\xi_k)^{\frac{1}{q}})^p}{(\triangle\xi_k)^{p-1}} = \sum_{k=0}^{n-1} \frac{\int_{\xi_k}^{\xi_{k+1}} |f|^p (\triangle\xi_k)^{\frac{p}{q}}}{(\triangle\xi_k)^{p-1}}$ but since $\frac{p}{q} = p - 1$ we have $\sum_{k=0}^{n-1} \frac{\int_{\xi_k}^{\xi_{k+1}} |f|^p (\triangle\xi_k)^{\frac{p}{q}}}{(\triangle\xi_k)^{p-1}} = \sum_{k=0}^{n-1} \int_{\xi_k}^{\xi_{k+1}} |f|^p = ||f||_p^p$ this gives the result. //

Exercise 134/21

Let g be an integrable function on $[0, 1]$. Find a bounded measurable function f such that $||f|| \neq 0$ and $\int fg = ||g||_1 ||f||_\infty$.

Proof. Let $f = sgn(g)$. Then $||f||_\infty = 1$ and $fg = |g|$. This gives that $F(f) = \int fg = \int |g| = ||g||_1 = ||g||_1 ||f||_\infty$. //

Now let $f \in L^\infty$, then $\forall\epsilon > 0$ show that $\exists g \in L^1$ such that $\int fg \geq (||f||_\infty - \epsilon)||g||_1$.

Proof. Let $E = \{x||f| \geq ||f||_\infty - \epsilon\}$. Let $g = \chi_E$. Then we have that $fg = 0$ off of E and $fg \geq ||f||_\infty - \epsilon$ on E. Thus we have $\int fg \geq (||f||_\infty - \epsilon)m(E) = (||f||_\infty - \epsilon)||g||_1$. //

Exercise 263/14

Let $(X, \mathcal{B}, \mu)$ be a measure space and $(X, \mathcal{B}_0, \mu_0)$ be its completion. Then f is measurable with respect to $\mathcal{B}_0$ iff there is a function g measurable with respect to $\mathcal{B}$ such that $f = g$ on the complement of a set $E \in \mathcal{B}$ such that $\mu(E) = 0$.

Proof. Let f be measurable in $\mathcal{B}_0$. Then $\forall\alpha \in \mathbb{Q}$, let $B_\alpha = \{x|f(x) \leq \alpha\}$, we have then that $B_\alpha \in \mathcal{B}_0$. Let $B_\alpha = B'_\alpha \cup A_\alpha$, where $B'_\alpha \in \mathcal{B}$ and $A_\alpha \subset C_\alpha$ with $C_\alpha \in \mathcal{B}$ and $\mu(C_\alpha) = 0$. Then for $\alpha < \beta$ we have $B'_\alpha \cup A_\alpha \subset B'_\beta \cup A_\beta$, and $B'_\alpha \setminus B'_\beta \in \mathcal{B}$. Further since $B'_\alpha \setminus B'_\beta = B'_\alpha \cap (X \setminus B'_\beta) \subset B'_\alpha \cap A_\beta \subset A_\beta \subset C_\beta$ and $\mu(C_\beta) = 0$ we have $\mu(B'_\alpha \setminus B'_\beta) = 0$. Then the collection $\{B_\alpha\}$ satisfies the condition of the proposition 11.10 so there is a function g measurable in $\mathcal{B}$ such that $f = g$ a.e.

Now let f be such that there is a function g measuable in $\mathcal{B}$ such that $f = g$ on $X \setminus E$ where $E \in \mathcal{B}$ and $\mu(E) = 0$. Let $\alpha \in \mathbb{R}$. Let $B'_\alpha \in \mathcal{B}$ such that

$B'_\alpha = \{x|g(x) \le \alpha\}$. Consider $B_\alpha = \{x|f(x) \le \alpha\}$. Let $C = B_\alpha \setminus B'_\alpha$. Then we have that $C \subset E$ and $B_\alpha = B'_\alpha \cup (B_\alpha \setminus B'_\alpha)$ and further $\mu(E) = 0$ so that $B_\alpha \in \mathcal{B}_0$.
//

Exercise 270/24: Let $(X, \mathcal{B})$ be a measure space and $\{\mu_n\}$ be a sequence of measures. such that $\mu_{n+1}(E) \ge \mu_n(E)$. Let $\mu = lim \mu_n$, then μ is a measure on $\mathcal{B}$.

Proof. Since $\mu_n(\emptyset) = 0$ we have that $\mu(\emptyset) = 0$. So we must show that $\mu(\bigcup E_k) = \sum \mu(E_k)$ for $\{E_k\}$ a disjoint collection. Since $\mu_n(E_k) \le \mu(E_k)$ we have $\mu(\bigcup E_k) \le \sum \mu(E_k)$. Now for any fixed l, we have that $\sum \mu_n(E_k) \ge \sum_{k=1}^{l} \mu_n(E_k) = \mu_n(\bigcup_{k=1}^{l} E_k)$. Taking limits on n gives us that $\mu(\bigcup E_k) \ge \sum_{k=1}^{l} \mu(E_k)$. However this was true for any l so we have that $\mu(\bigcup E_k) \ge \sum \mu(E_k)$.
//

Exercise 270/25 Give and example of a sequence $\{\mu_n\}$ of measures such that the set function $\mu = lim \mu_n$ is not a measure.

Proof. Let $\{A_k\}$ be a countable collection of disjoint sets. Let $X = \bigcup A_k$. Let $\mathcal{B}$ be the smallest $\sigma-$algebra containing $\{A_k\}$. Let $\mu_n(A_k) = \frac{1}{k}$ and for $B \in \mathcal{B}$ let $\mu_n(B) = \sum_{k=1}^{\infty} \mu_n(B \cap A_k)$. Then μ_n are each measures. Let $\mu = lim \mu_n$. We have then that $\mu_n(X) = \infty$ for all n so $\mu(X) = \infty$. However $\mu(\bigcup A_k) = \sum \mu(A_k) = \sum lim \mu_n(A_k) = \sum 0 = 0$, so μ is not countably additive. //

Exercise 270/26 Let $(X, \mathcal{B})$ be a measure space and $\{\mu_n\}$ be a sequence of meausres such that $\mu_{n+1}(E) \le \mu_n(E)$ and $\mu = lim \mu_n$ such that $\mu(X) < \infty$. Then show μ is a measure.

Proof. Since $\mu_n(\emptyset) = 0$ we have that $\mu(\emptyset) = 0$. So we must show that $\mu(\bigcup E_k) = \sum \mu(E_k)$ for $\{E_k\}$ a disjoint collection. Since $\mu_n(E_k) \ge \mu(E_k)$ we have that $\mu(\bigcup E_k) \ge \sum \mu(E_k)$. So we need to show that $\mu(\bigcup E_k) \le \sum \mu(E_k)$. Without loss of generality we can assume $X = \bigcup E_k$. Now defined $A_l = \bigcup_{k=l}^{\infty} E_k$. Then we have that $A_{l+1} \subset A_l$ and $\bigcap A_l = \emptyset$. By proposition for each n, $\mu_n(\bigcap A_l) = \lim_l \mu_n(A_l) = 0$ so $\lim_l \mu(A_l) = 0$. Now we have that $E_k = A_k \setminus A_{k+1}$. This gives that $\mu_n(E_k) = \mu_n(A_k \setminus A_{k+1}$. Since μ_n is a measure, we have that $\mu_n(E_k) \ge \mu_n(A_k) - \mu_n(A_{k+1})$ and passing to the limit we have $\mu(E_k) \ge \mu(A_k) - \mu(A_{k+1})$. Then we have that $\sum \mu(E_k) \ge \sum \mu(A_k) - \mu(A_{k+1}) = \mu(A_0) - lim_l \mu(A_l) = \mu(\bigcup E_k) - 0 = \mu(\bigcup E_k)$. //

15. Midterm 2005

1. Let X, Y be nonempty sets, let $\mathcal{A}$ be a $\sigma-$algebra of subsets of X, and f be a function from X to Y. Define $\mathcal{E}$ to be the family of all subsets E of Y such that $f^{-1}[E] \in \mathcal{A}$. Prove that $\mathcal{E}$ is a $\sigma-$algebra of subsets of Y.

Proof. Let $A \in \mathcal{E}$ then $f^{-1}(A^c) = f^{-1}(A)^c \in \mathcal{A}$ which give that $A^c \in \mathcal{E}$. Likewise let $\{A_\alpha\} \subset \mathcal{E}$, then $f^{-1}(A_\alpha) \in \mathcal{A}$ which gives that $\bigcup_\alpha f^{-1}(A_\alpha) = f^{-1}(\bigcup_\alpha A_\alpha) \in \mathcal{A}$, thus $\bigcup_\alpha A_\alpha \in \mathcal{E}$. Now $f^{-1}(Y) = X$ so we have that $Y \in \mathcal{E}$ and so $\emptyset \in \mathcal{E}$. This gives that $\mathcal{E}$ is a $\sigma-$algebra. //

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function whose derivative f' is bounded. That is there exists a positive finite constant K such that $|f'(x)| \leq K$ for all $x \in \mathbb{R}$. In this problem, you may assume all the standard properties of Lebesgue out measure m^*.

(a) Show that if E is any subset of $\mathbb{R}$ then $m^*(f[E]) \leq Km^*(E)$ where $f[E]$ is the image of E under f. [Suggestion: Begin by doing it for E an open interval (a, b), using the Mean Value Theorem from calculus.]

(b) Deduce that if E is a set of (outer) measure 0 in $\mathbb{R}$, $m(E) = m^*(E) = 0$, then $f[E]$ is a measurable set.

Proof. (a)First consider $E = I = (a, b)$ an open inteval. We may suppose $m^*(I) < \infty$. Let $\lambda = inf\{f(x)|x \in I\}$ and $\mu = sup\{f(x)|x \in I\}$. If $s, t \in I$ then $f(t)-f(s) = f'(c)(t-s)$ for some $c \in I$, by the Mean Value Theorem if $s \neq t$, or for any $c \in I$ if $s = t$. Let $|f'(c)| \leq K$, so $f(t) - f(s) \leq |f'(c)||t-s| \leq K(b-a)$. For fixed $s \in I$, $f(t) \leq K(b-a) + f(s)$ for each $t \in I$ and by taking supremum over t we have $\mu \leq K(b-a) + f(s)$. Then μ is finite and $\mu - K(b-a) \leq f(s)$ for each $s \in I$. Taking infimum over s we have then that $\mu - K(b-a) \leq \lambda$ so λ is finite, and we have that $\mu - \lambda \leq K(b-a)$. The since $f[I] \subset [\lambda, \mu]$ we have $m^*(f[I]) \leq m^*([\lambda, \mu]) = \mu - \lambda \leq K(b-a) = Kl(I) = Km^*(I)$. Now suppose $E \subset \mathbb{R}$ is arbitrary. We may suppose $m^*(E) < \infty$. Given $\epsilon > 0$ then we have that there is $\{I_n\}$ such that $E \subset \bigcup_n I_n$ and $\sum l(I_n) < m^*(E) + \frac{\epsilon}{K}$. Then $f[E] \subset \bigcup f[I_n]$ so we have $m^*(f[E]) \leq \sum m^*(f[I_n]) \leq \sum Kl(I_n) = K\sum l(I_n) < K(m^*(E) + \frac{\epsilon}{K}) = Km^*E + \epsilon$ which gives $m^*(f[E]) \leq Km^*(E)$.

(b) If $m^*(E) = 0$ then we have that $m^*(f[E]) \leq Km^*(E) = 0$ thus $f(E)$ is measurable. //

3. Reacll that if f is a nonnegative measurable function on $\mathbb{R}$ and f is integrable then f is finite almost everywhere, that is, $m(\{x|f(x) = \infty\}) = 0$. Suppose that $\{u_n\}$ is an infinite sequence of nonnegative measurable functions on $\mathbb{R}$ such that $\sum \int u_n < \infty$. Show that the series $\sum u_n$ converges almost everywhere, that is, the series $\sum u_n(x)$ converges for all x except for a set of measure 0.

Proof. Let $f = \sum u_n$ be a nonngeative measurable function. By (a corollary to) the Monotone Convergence Theorem, $\int f = \sum \int u_n < \infty$. Thus f is integrable, let $A = \{x|f(x) = \infty\}$ so $m(A) = 0$. If $x \in \mathbb{R} \setminus A$ then $\sum u_n(x) = f(x) < \infty$, so the series $\sum u_n(x)$ converges. //

16. Final 2005

1. Let f be a bounded real-valued function on the square $[0,1] \times [0,1]$. Let $f_x(y) = f(x,y)$, $f^y(x) = f(x,y)$ and suppose that:

(i) For each number $x \in [0,1]$ f_x is continuous
(ii) For each rational $y \in [0,1]$then f^y is Lebesgue measurable

Show that $g(y) = \int_{[0,1]} f(x,y)dx$ is a well-defined continuous function on $[0,1]$

Proof. We have by (ii) that $g(q)$ is well-defined for each rational $q \in [0,1]$. Now let $y \in [0,1]$ and $r_n \to y$ where r_n is rational. Then we have that $f(x,r_n) \to f(x,y)$ since $g(x,\cdot)$ is continuous, and so we have that $f(x,y)$ is a bounded measureable function, hence integrable and so $g(y)$ is well-defined. Now let $y \in [0,1]$ and let $\{y_n\}$ be any sequence $y_n \to y$. Then we have that $g(y_n) = \int f(x,y_n) \to \int f(x,y)$ by bounded convergence so g is continuous at y. //

2. This problem studies the value of $\lim\limits_{p\to 0^+} ||f||_p$ for a lebegue measurable function f on $[0,1]$

a) Show that if f is measurable on $[0,1]$ and $0 < p_1 < p_2 < \infty$ then $||f||_{p_1} \leq ||f||_{p_2}$ (Hint: $||f||_{p_1}^{p_1} = \int |f|^{p_1} \cdot 1$).

Proof. If $||f||_{p_2} = \infty$ then we are done, so without loss of generality assume $||f||_{p_2} < \infty$. Let $p = \frac{p_2}{p_1} > 1$. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we have that $||f||_{p_1}^{p_1} = \int |f|^{p_1} = \int |f|^{p_1} * 1 \leq ||f^{p_1}||_p * ||1||_q = ||f^{p_1}||_p$ by Hölder's inequality. Now $||f^{p_1}||_p = (\int (|f|_1^p)^p)^{\frac{1}{p}} = ((\int |f|_2^p)^{\frac{1}{p_2}})_1^p = ||f||_{p_2}^{p_1}$ which gives that $||f||_{p_1} \leq ||f||_{p_2}$ //

b) Suppose $0 < p_1 < p_2 < \infty$. Give an example of a measurable function f on $[0,1]$ such that $||f||_{p_1} < \infty$ but $||f||_{p_2} = \infty$. (Of course, the formula for you answer will involve p_1 and/or p_2)

Proof. Let $f(x) = \frac{1}{x^{\frac{1}{p_2}}}$ then we have that $||f||_{p_1}^{p_1} = \int \frac{1}{x^{\frac{p_1}{p_2}}}$ which is finite since $\frac{p_1}{p_2} < 1$. However $||f||_{p_2}^{p_2} = \int \frac{1}{x^{\frac{p_2}{p_2}}} = \int \frac{1}{x} = \infty$. //

c) Show that if φ is a (strictly) positive simple function on $[0,1]$ then $\lim\limits_{p\to 0^+} ||\varphi||_p = e^{\int \log(\varphi)}$ (Suggestion: Apply l'Hopital's rule to $log||\varphi||_p$)

Proof. Given $\varphi = \sum_{i=1}^n a_i \chi_{A_i}$ a strictly positive simple function, that is $\sum \mu(A_i) = 1$ and $a_i > 0$ then we compute $\lim\limits_{p\to 0^+} \log ||\varphi||_p = \lim\limits_{p\to 0^+} \log((\sum a_i^p \mu(A_i))^{\frac{1}{p}}) = \lim\limits_{p\to 0^+} \frac{\log((\sum a_i^p \mu(A_i))}{p}$. Here evaluating the limit gives an indeterminate form, so using L'Hopitals rule we have $\lim\limits_{p\to 0^+} \log ||\varphi||_p = \lim\limits_{p\to 0^+} \frac{\log((\sum a_i^p \mu(A_i))}{p} = \lim\limits_{p\to 0^+} \frac{(\sum a_i^p \log(a_i)\mu(A_i)}{(\sum a_i^p \mu(A_i))}$. Now taking the limit we arrive at $\frac{(\sum \log(a_i)\mu(A_i)}{(\sum \mu(A_i))} = \sum \log(a_i)\mu(A_i) = \int \log(\varphi)$. So we have that $\lim\limits_{p\to 0^+} \log ||\varphi||_p = \int \log(\varphi)$ and after exponentiating we obtain $\lim\limits_{p\to 0^+} ||\varphi||_p = e^{\int \log(\varphi)}$. //

d) Show that if f is measurable on $[0,1]$, then for all p for which $|f|^p$ is integrable, $e^{\int \log|f|} \leq ||f||_p$ where integrals may be infinite, $log 0 = -\infty, e^{\infty} = \infty$ and $e^{-\infty} = 0$.

Proof. We have that $e^{\int \log(|f|)^p} = e^{p\int \log(|f|)} = e^{\int \log(|f|^p)} \leq \int e^{\log|f|^p} = \int |f|^p = ||f||_p^p$ which gives that $e^{\int \log(|f|)} \leq ||f||_p$ //

e) Show that if f is bounded measurable on $[0,1]$ (that is $f \in L^{\infty}$), then $\lim\limits_{p\to 0^+} ||f||_p = e^{\int \log|f|}$.

Proof. Let $\varphi_n \searrow |f|$ be a sequence of strictly positive simple functions. Then we have that $|f|^p \leq \varphi_n^p$ and $\log\varphi_n \searrow \log|f|$, so we have by monotone convergence that $\int \log(\varphi_n) \searrow \int \log|f|$. For each n we have that $\overline{\lim\limits_{p\to 0^+}} ||f||_p \leq \overline{\lim\limits_{p\to 0^+}} ||\varphi_n||_p = e^{\int \log(\varphi_n)}$. If we let $n \to \infty$ then we have that $\overline{\lim\limits_{p\to 0^+}} ||f||_p \leq e^{\int \log|f|}$ and so by (d) we have that $\lim\limits_{p\to 0^+} ||f||_p = e^{\int \log|f|}$. //

3. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be finite measure spaces (that is $\mu(X) < \infty, \nu(Y) < \infty$). Let $\mathcal{R} = \{A \times B | A \in \mathcal{A}, B \in \mathcal{B}\}$ and let $\mathcal{R}'$ consist of finite unions of sets in $\mathcal{R}$. Let $\mathcal{A}\times\mathcal{B}$ denote $\sigma(\mathcal{R})$ For $E \subset X \times Y$ and $x \in X$ let $E_x = \{y \in Y | (x,y) \in E\}$. Take as given that there is a unique measure λ on $\mathcal{A}\times\mathcal{B}$ that satisfies $\lambda(A\times B) = \mu(A)\cdot\nu(B)$ for each $A \times B \in \mathcal{R}$. Our goal is to study the equation

$$(*) \quad \lambda(E) = \int_X \nu(E_x) d\mu$$

(a) Show that if $E \in \mathcal{A}\times\mathcal{B}$ and $x \in X$ then $E_x \in \mathcal{B}$

Proof. Let $\mathcal{C} = \{E \in \mathcal{A}\times\mathcal{B} | E_x \in \mathcal{B}\}$. We have that $\forall E \in \mathcal{R}, E = A \times B$ then $E_x \in \{B, \emptyset\}$, so we have that $E \in \mathcal{C}$ so $\mathcal{R} \subset \mathcal{C}$. We have also then that $X \times Y \in \mathcal{C}$. Now, given $E \in \mathcal{C}$ then we have that $E_x \in \mathcal{B}$ which gives that $(E^c)_x) = (E_x)^c \in \mathcal{B}$ so $E^c \in \mathcal{C}$. Now given $\{E_n\}$ a disjoint collection in $\mathcal{C}$, then we have that $\forall n, (E_n)_x \in \mathcal{B}$ which gives that $\bigcup (E_n)_x = (\bigcup E_n)_x \in \mathcal{B}$ thus $\bigcup E_n \in \mathcal{C}$ and $\mathcal{C}$ is a $\sigma-$algebra, and since $\mathcal{R} \subset \mathcal{C}$ we have that $\mathcal{A}\times\mathcal{B} \subset \mathcal{C}$. //

(b) Let $\mathcal{R}_\sigma$ consist of those sets that are finite or countable unions of sets in $\mathcal{R}$ (or $\mathcal{R}'$). Assuming that (*) holds for every $E \in \mathcal{R}'$, prove that (*) holds for every $E \in R_\sigma$

Proof. Let $E \in \mathcal{R}_\sigma$ and write $E = \bigcup E_n$ a disjoint union of members of $\mathcal{R}'$. For any x we have $E_x = \bigcup (E_n)_x$ and this union is also disjoint, so we have $\nu(E_x) = \sum \nu((E_n)_x)$. so by monotone convergence $\int \nu(E_x) = \int \sum \nu((E_n)_x) = \sum \int \nu((E_n)_x) = \sum \lambda(E_n) = \lambda(\bigcup E_n) = \lambda(E)$. //

(c) With the same assumption as in part (b), let $\{E_n\}$ be an infinite sequence in R_σ such that $E_n \supset E_{n+1}$ for every n, and let $E = \bigcap E_n$. Prove that (*) holds for E.

Proof. Given $E_n \searrow E$ then $\lambda(E_n) \searrow \lambda(E)$ since we are finite spaces. This gives then for each x that $(E_n)_x \searrow E_x$, and so $\nu((E_n)_x) \searrow \nu(E_x)$. So we have that $\int \nu(E_x) d\mu = \lim \int \nu((E_n)_x) d\mu = \lim \lambda(E_n) = \lambda(E)$. //

17. Prelim Material

17.1. Syllabus. Lebesgue Measure and Integration in $\mathbb{R}^n$

a) Lebesgue measureable sets, Borel sets
b) Measurable functions, modes of convergence (uniform, pointwise, a.e., in measure).
c) Lebesgue integral, convergence theorems
d) Functions of bounded variation, absolutely continuous functions, differentiation, Lebesgue decomposition of measures
e) Relationship between Riemann and Lebesgue integration

General Measure and Integration

a) Measurable spaces, measurable functions
b) Measure spaces, Caratheodory's theorem
c) Integration, convergence theorems
d) Signed measures, Complex measures, Hahn and Jordan decompositions
e) Radon-Nikodym theorem
f) Basic inequalities(Cauchy-Schwartz, Jensen, Hölder, Minkowski)
g) L^p spaces completeness duality
h) Riesz Representation theorem
i) Fubini-Tonelli theorem

17.2. Things to prove. I have annotated page number references to where the proofs of these results can be found.

(1) $\int_A f d\mu < \epsilon$ for $\mu(A) < \delta$ 57
(2) Fatou Lemma $\iff$ Monotone Convergence Theorem 14
(3) Egoroff 9
(4) $\frac{d}{dt}\int f(x,t)dx = \int \frac{\partial f}{\partial t} f(x,t)dx$ 53
(5) Modes of Convergence Section 31
(6) Young $\Rightarrow$ Hölder $\Rightarrow$ Minkowski 26
(7) BV $\int |f'(x)| \leq T_a^b f$ and equality if a. cts. 78
(8) When is are Lebegue integrable functions Riemann integrable 66
(9) $E + F$ contains an interval 35
(10) L^p is complete 28
(11) $f * g$ is continuous for $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$. 35
(12) $\lim_{p\to\infty} ||f||_p \to ||f||_\infty$ 69
(13) When is $L^r(X,\mu) \subset L^p(X,\mu)$ 29
(14) For $\{f_n\} \subset L^p$ then $||f_n||_p \to ||f||_p$ iff $||f_n - f||_p \to 0$ 28.
(15) $\lim_{p\to 0^+} ||f||_p = e^{\int (log|f|)}$ for $f \in L^r(\mathbb{R}$ for some r. 69
(16) $\int f_n \cdot g \to \int f \cdot g$ for $\{f_n\} \subset L^p$ and $g \in L^q$, with $\frac{1}{p} + \frac{1}{q} = 1$ 70.
(17) $||f * g||_p \leq ||f||_p ||g||_1$ for $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ 56
(18) General Minkowski 68

17.3. Things to Know.

(1) Radon-Nikodym
(2) Fubini/Tonelli
(3) Riesz Representation
(4) Vitali Covering and the Vitali Covering Lemma
(5) Jensen's inequality

17.4. **Technqiues/Rules of thumb.**

(1) All Littlewood principles
(2) For $f \in L^1$ you can obtain a step function ψ with $\int f = \int \psi + \epsilon$.
(3) To show bounded variation or absolutely continuous, try bounding the derivative
(4) For dominated sequences you can gain almost uniform convergence
(5) To show that $\int f$ has some property, first show the property when f is simple, then use convergence theorems and a simple approximation of f to get for $\int f$
(6) Write your function f as a sequence of functions, (like $f' = \lim_{n\to\infty} f_n$ where $f_n(x) = n[f(x + \frac{1}{n})) - f(x)])$
(7) Simple functions are dense in L^q
(8) On a finite space, $1 \in L^p$ for any p so you can pretty much Hölder whenever you want.
(9) Write $\mu(A) = \int_A 1 d\mu$
(10) $\mu << |\mu|$

17.5. **Problems and solutions.**

2004

1. State clearly and completely all major results that you use and cite in your work.

2. Prove or disprove:

a) If a sequence of Lebesgue-measureable functions on $[0,1]$ converges a.e., then it converges in measure

b) If a sequence of Lebesgue-measureable functions $\{f_n\}$ converges to f in $L^1([0,1])$ then $f_n \to f$ a.e. on $[0,1]$

c) If $f \in L^1(X, \mathcal{F}, \mu)$ and $f_n \in L^1(X, \mathcal{F}, \mu)$ for each $n \in \mathbb{N}$ then if f_n converges a.e. and $\int |f_n| \to \int |f|$ then $f_n \to f$ in $L^1(X, \mathcal{F}, \mu)$

d) If $f \in L^1(\mathbb{R})$ and f is uniformly continuous on $\mathbb{R}$ then $f \in C_0(\mathbb{R})$.

e) If $f \in L^1(X, \mathcal{F}, \mu)$ and $\mathcal{G} \subset \mathcal{F}$ a $\sigma-$algebra. Then there exists a $\mathcal{G}-$measurable function g such that for each $A \in \mathcal{G}$ we have $\int_A g d\mu = \int_A f d\mu$.

3. Suppose f is a non negative integrable function on a space $(X, \mathcal{F}, \mu)$.

a) $lim_{\lambda \to \infty} \lambda \mu(f \geq \lambda) = 0$

b) Produce a non-negative Lebesgue measureable function f on $[0,1]$ such that above holds but $f \notin L^1([0,1])$.

c) Suppose that μ is a finite measure, such that f is a non-negative $\mathcal{F}-$measurable function. Prove $f \in L^1(X, \mathcal{F}, \mu) \iff \sum_{n=1}^{\infty} \mu(f > n) < \infty$.

4. Let $p \in [1, \infty)$ prove that if $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ then $f * g(x) = \int f(x-y)g(y)dy$ is well defined a.e. and $f * g \in L^p(\mathbb{R})$ such that $||f * g||_{L^p} \leq ||f||_{L^p} ||g||_{L^1}$.

5. Let $(\mathbb{R}, \mathcal{M} \cap (0,1), m)$ be the Lebesgue measure space on $\mathbb{R}$. For a space $(X, \mathcal{F}, \mu)$ consider a $\mathbb{R}$ valued $\sigma(\mathcal{M} \times \mathcal{F})-$measurable function f on $(0,1) \times X$ such that

i For $t \in (0,1)$ then $f(t, \cdot) \in L^1(\mu)$

ii For $x \in X$, then $f(\cdot, x)$ is differentiable on $(0,1)$

iii There is a function $g \in L^1(\mu)$ such that for $s \in (0,1)$ and $x \in X$ we have $|\frac{\partial f}{\partial t}(s, x)| \leq g(x)|$.

Then define $\varphi : (0,1) \to \mathbb{R}$ by $\varphi(t) = \int f(t, x) d\mu$. Then

a) Prove that φ is differentiable at every point of $(0,1)$ and that for $s \in (0,1)$ we have $\varphi'(s) = \int \frac{\partial f}{\partial t}(s, x) d\mu$.

b) Determine whether φ is absolutely continuous $(0,1)$.

2004 Solutions

1. State clearly and completely all major results that you use and cite in your work.

2. Prove or disprove:

a) If a sequence of Lebesgue-measureable functions on $[0,1]$ converges a.e., then it converges in measure

Proof. Prove. This is so as we are in a finite space, thus Egoroff's theorem gives convergence almost everywhere gives convergence almost uniform and this in turn gives convergence in measure. //

b) If a sequence of Lebesgue-measureable functions $\{f_n\}$ converges to f in $L^1([0,1])$ then $f_n \to f$ a.e. on $[0,1]$

Proof. Disprove. Consider $(\mathbb{R}, \mathcal{M}, m)$. Let $\{I_n\}$ be a enumeration of subintervals of the unit interval by: $I_1 = [0,1], I_2 = [0, \frac{1}{2}], I_3 = [\frac{1}{2}, 1], I_3 = [0, \frac{1}{3}], I_4 = [\frac{1}{3}, \frac{2}{3}], I_5 = [\frac{2}{3}, 1], \ldots$. Let $f_n = \chi_{I_n}$. Then we have that f_n does not converge $\mu-$a.e. However for $n > \frac{m(m+1)}{2}$ we have that $\int f_n < \frac{1}{m}$ which gives that $\{f_n\}$ does converge in mean or order 1 to $f = 0$. //

c) If $f \in L^1(X, \mathcal{F}, \mu)$ and $f_n \in L^1(X, \mathcal{F}, \mu)$ for each $n \in \mathbb{N}$ then if f_n converges a.e. and $\int |f_n| \to \int |f|$ then $f_n \to f$ in $L^1(X, \mathcal{F}, \mu)$

Proof. This is true. Let $g_n = |f_n - f|$ and $h_n = |f_n| + |f|$. Then $g_n \to 0\mu-$a.e. Since $f \in L^1(X, \mathcal{F}, \mu)$ we have that $2f \in L^1(X, \mathcal{F}, \mu)$. Let $h = 2f$. Then we have that $\int h_n \to \int h$ by hypothesis and $|g_n| \leq h_n$ by the triangle inequality, hence by Lebesgue dominated convergence theorem we have that $\int g_n \to \int g = 0$ //

d) If $f \in L^1(\mathbb{R})$ and f is uniformly continuous on $\mathbb{R}$ then $f \in C_0(\mathbb{R})$. [7]

e) If $f \in L^1(X, \mathcal{F}, \mu)$ and $\mathcal{G} \subset \mathcal{F}$ a $\sigma-$algebra. Then there exists a $\mathcal{G}-$measurable function g such that for each $A \in \mathcal{G}$ we have $\int_A g d\mu = \int_A f d\mu$.

Proof. Let $X = \mathbb{N}$ and $\mathcal{F} = \wp(X)$ with μ the counting measure. Let $\mathcal{G} = \{\emptyset, \mathbb{N}\}$. Then for $0 \neq f \in L^1(\mu)$, any such g that is $\mathcal{G}-$measurable must be constant. However then $\int_X g = \infty$ and since $f \int L^1$ we cannot have $\int_X f d\mu = \int_X g d\mu$. //

3. Suppose f is a non negative integrable function on a space $(X, \mathcal{F}, \mu)$.

a) $lim_{\lambda \to \infty} \lambda \mu(f \geq \lambda) = 0$

Proof. We have that $\lambda \mu(f \geq \lambda) = \lambda \int \chi_{f \geq \lambda} d\mu$. Now we have also that $\lambda \chi_{f \geq \lambda} \leq f$ and since $f \in L^1(\mu)$ we have that $lim \lambda \int \chi_{f \geq \lambda} = \int lim \lambda \chi_{f \geq \lambda} d\mu = \int 0 d\mu = 0$. //

b) Produce a non-negative Lebesgue measureable function f on $[0,1]$ such that above holds but $f \notin L^1([0,1])$.

[7] Due to some argument as to the exact definition of C_0 on this prelim (Different references define this differently, and the question is which one was used on this particular years prelim), I have omitted the proof.

Proof. Let $f = \frac{1}{x(1-lnx)}$ then $f \notin L^1(\mu)$ as $\int_0^1 \frac{dx}{x(1-lnx)} = lim_{b\to 0^+}\int_0^1 \frac{-du}{u} = lim_{b\to 0^+}[-ln|1-lnx|]_b^1 \to \infty$. However we have that $lim\lambda\mu(f \geq \lambda) = lim_{a\to infty}e^{-a}(\frac{e^a}{1+a}) = lim\frac{1}{1+a} = 0$. //

c) Suppose that μ is a finite measure, such that f is a non-negative $\mathcal{F}-$measurable function. Prove $f \in L^1(X, \mathcal{F}, \mu) \iff \sum_{n=1}^{\infty}\mu(f > n) < \infty$.

Proof. Let $f \in L^1(\mu)$ then $\sum_{n=1}^{\infty}\mu(f>n) = \sum_{n=1}^{\infty}\int\chi_{f>n} = \int\sum_{n=1}^{\infty}\chi_{f>n} \leq \int f < \infty$. Let $\sum_{n=1}^{\infty}\mu(f>n) < \infty$. Then we have that $f(x) \leq \sum_{n=1}^{\infty}\chi_{f>n} + 1$. Thus we have $\int f \leq \int(\sum_{n=1}^{\infty}\chi_{f>n} + 1)d\mu = \int\sum_{n=1}^{\infty}\chi_{f>n} + \int 1 d\mu = \sum_{n=1}^{\infty}\mu(f>n) + \mu(X) < \infty$ as μ is a finite measure. //

4. Let $p \in [1, \infty)$ prove that if $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ then $f * g(x) = \int f(x-y)g(y)dy$ is well defined a.e. and $f * g \in L^p(\mathbb{R})$ such that $||f*g||_{L^p} \leq ||f||_{L^p}||g||_{L^1}$.

Proof.

$$\begin{aligned} ||f*g||_p &= \left(\int_X (f*g(x))^p dx\right)^{\frac{1}{p}} \\ &= \left(\int\left(\int f(x-y)g(y)dy\right)^p dx\right)^{\frac{1}{p}} \\ \text{By General Minkowski} &\leq \int\left(\int(f(x-y)g(y))^p dx\right)^{\frac{1}{p}} dy \\ &= \int g(y)||f||_p dy \\ &= ||f||_p||g||_1 \end{aligned}$$

This also shows that $f * g$ is well defined a.e. //

5. Let $(\mathbb{R}, \mathcal{M} \cap (0,1), m)$ be the Lebesgue measure space on $\mathbb{R}$. For a space $(X, \mathcal{F}, \mu)$ consider a $\mathbb{R}$ valued $\sigma(\mathcal{M} \times \mathcal{F})-$measurable function f on $(0,1) \times X$ such that

i For $t \in (0,1)$ then $f(t, \cdot) \in L^1(\mu)$

ii For $x \in X$, then $f(\cdot, x)$ is differentiable on $(0,1)$

iii There is a function $g \in L^1(\mu)$ such that for $s \in (0,1)$ and $x \in X$ we have $|\frac{\partial f}{\partial t}(s,x)| \leq g(x)|$.

Then define $\varphi : (0,1) \to \mathbb{R}$ by $\varphi(t) = \int f(t,x)d\mu$. Then

a) Prove that φ is differentiable at every point of $(0,1)$ and that for $s \in (0,1)$ we have $\varphi'(s) = \int \frac{\partial f}{\partial t}(s,x)d\mu$.

Proof. Fix $s \in (0,1)$. Let $h_n(x) = n[f(s+\frac{1}{n}, x) - f(s,x)]$. Now $h_n(x) \to \frac{\partial f}{\partial t}(s,x)$. Let $h(x) = \frac{\partial f}{\partial t}(s,x)$. Further we have that $|h_n(x)| = |\frac{\partial f}{\partial t}(s', x)| \leq g(x) \in L^1(\mu)$ by the mean value theorem. This gives then that $\int h_n \to \int h$. However we have now that $\varphi'(s) = lim_{n\to 0}n[\int f(s+\frac{1}{n}, x) - \int f(s,x)] = lim_{n\to 0}\int h_n = \int h = \int \frac{\partial f}{\partial t}(s,x)$. //

b) Determine whether φ is absolutely continuous $(0,1)$.

Proof. Part (a) gives that φ is absolutely continuous as it is an anti-derivative. //

2003

1. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let f be a realvalued measureable function. Then for $p \in (0, \infty)$ let $\varphi(p) = \int |f|^p d\mu = ||f||_p^p$. Let $E = \{p|\varphi(p) < \infty\}$. Assume that $||f||_\infty < \infty$ then:

a) Prove that either $E \neq \emptyset$ or E is an unbounded interval of $(0, \infty)$ and that if $\varphi(p) > 0$ for some (equivalently all) $p > 0$ and $E \neq \emptyset$ then $\log \varphi$ is convex on E.

b) Prove that if $E \neq \emptyset$ then φ is a continuous function.

c) Is E necessarily open? Closed?

2. Prove: for $p \in [1, \infty)$ then if $f \in L^P(\mathbb{R}, \mathcal{M}, m)$ and $g \in L^1(\mathbb{R}, \mathcal{M}, m)$ then $||f * g||_p \leq ||f||_p ||g||_1$.

3. Let $(X, \mathcal{F}, \mu)$ be a finite measure space.

a) Prove or disprove: If a sequence $\{f_n\}$ of real valued $\mathcal{F}-$measureable functions on X converge $\mu-$a.e. then $\{f_n\}$ converges in measure.

b) Prove or disprove: If a sequence $\{f_n\}$ of real valued $\mathcal{F}-$measureable functions on X converge in measure then $\{f_n\}$ converges $\mu-$almost everywhere.

c) Prove or disprove: If a sequence $\{f_n\}$ of real valued $\mathcal{F}-$measureable functions on X is Cauchy in $L^1(\mu)$ then $\{f_n\}$ converges in measure

4. Let $(X, \mathcal{F}, \mu)$ be a measure space.

a) Prove that if $f \in L^1(\mu)$ then for every $\epsilon > 0$ there is $\delta > 0$ such that $\mu(A) < \delta$ then $\int_A |f| d\mu < \epsilon$.

b) A sequence $\{f_n\}$ is said to have uniformaly absolutely continuous intergrals if for $\epsilon > 0$ there is $\delta > 0$ such that $\mu(A) < \delta$ then $\int_A |f_n| < \epsilon$ for each n. Suppose $\mu(X) < \infty$ and $\{f_n\} \to f$ has uniformaly absolutely continuous integrals. Then $f_n \to f$ in $L^1(\mu)$.

c) Show that (b) imples the Lebesgue Dominated Convergence Theorem

5. Let μ be a signed measure on $\mathcal{F}$ a $\sigma-$algebra. Prove by applying Radon-Nikodym that there is a unique $\mathcal{F}-$measureable function h such that $|h(x)| = 1\mu-$a.e. and $\mu(A) = \int_A h d|\mu|$ where $|\mu|$ is the total variation measure.

2003 Solutions

1. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let f be a realvalued measureable function. Then for $p \in (0, \infty)$ let $\varphi(p) = \int |f|^p d\mu = ||f||_p^p$. Let $E = \{p|\varphi(p) < \infty\}$. Assume that $||f||_\infty < \infty$ then:

a) Prove that either $E \neq \emptyset$ or E is an unbounded interval of $(0, \infty)$ and that if $\varphi(p) > 0$ for some (equivalently all) $p > 0$ and $E \neq \emptyset$ then $\log \varphi$ is convex on E.

Proof. We first show that E is connected by showing that if $r, s \in E$ then $(r, s) \subset E$. This is so by Hölders inequality. If we let $r \leq p \leq s$, then let $A = \{x|f(x) \geq 1\}$ can split $\int |f|^p = \int_A |f|^p + \int_{A^c} |f|^p \leq \int_A |f|^s + \int_{A^c} |f|^r < \infty$. Now we have that since $f \in L^\infty$ and $\lim_{p\to\infty} ||f||_p = ||f||_\infty$, that if $p \in E$ then $[p, \infty) \subset E$. This gives then that E must be an unbounded interval as when $p_0 = \inf(E)$ then we have that $[p, \infty) \subset E$ for $p > p_0$ thus $E = (p_0, \infty)$ or $[p_0, \infty)$. Now to show that $\log \varphi$ is convex on E. We note that for $\lambda \in [0, 1]$, and $x, y \in E$ we have that $\varphi(\lambda x + (1 - \lambda)y) = \int |f|^{\lambda x}|f|^{(1-\lambda)y}$. Now $|f|^{\lambda x} \in L^{\frac{1}{\lambda}}$ and $|f|^{(1-\lambda)y} \in L^{\frac{1}{1-\lambda}}$. This gives that by Hölder's inequality we have $\varphi(\lambda x+(1-\lambda)y) \leq ||f^{x\lambda}||_{\frac{1}{\lambda}}||f^{(1-\lambda)y}||_{\frac{1}{1-\lambda}} = \left(\int |f|^x\right)^\lambda \left(\int |f|^y\right)^{1-\lambda} = \varphi(x)^\lambda \varphi(y)^{1-\lambda}$. Now since log is monotonic we have that $\log \varphi(\lambda x + (1 - \lambda)y) \leq \log(\varphi(x)^\lambda \varphi(y)^{1-\lambda}) = \log(\varphi(x)^\lambda) + \log(\varphi(y)^{1-\lambda}) = \lambda \log(\varphi(x)) + (1 - \lambda) \log(\varphi(y))$. //

b) Prove that if $E \neq \emptyset$ then φ is a continuous function.

Proof. Since $\log(\varphi)$ is a convex function then it must also be continuous, and so we have that $\varphi = e^{\log(\varphi)}$ is also continuous. //

c) Is E necessarily open? Closed?

Proof. E does not necessarily need to be closed as $f(x) = \frac{1}{x}$ is in $L^p([1, \infty])$ for $p > 1$ but not in $L^1([1, \infty])$. E does however need to be open. We have that $\varphi(E^c) = \infty$. So consider $(0, \infty) \subset (0, \infty]$ in the range. We have then that $(0, \infty) = \varphi(E) \cup ((0, \infty) \setminus \varphi(E))$. Now taking inverse images $\varphi^{-1}(0, \infty) = \varphi^{-1}(\varphi(E)) \cup (\varphi^{-1}((0, \infty) \setminus \varphi(E)) = E \cup (\varphi^{-1}(0, \infty) \setminus E)$. However as we have $\varphi(E)^c = \infty$ we have then that $(\varphi^{-1}(0, \infty) \setminus E) = \emptyset$ which gives that $\varphi^{-1}(0, \infty) = E$ so E is open as φ is continuous. //

2. Prove: for $p \in [1, \infty)$ then if $f \in L^P(\mathbb{R}, \mathcal{M}, m)$ and $g \in L^1(\mathbb{R}, \mathcal{M}, m)$ then $||f * g||_p \le ||f||_p ||g||_1$.

Proof.

$$\begin{aligned}
||f * g||_p &= \left(\int_X (f * g(x))^p dx\right)^{\frac{1}{p}} \\
&= \left(\int \left(\int f(x-y)g(y)dy\right)^p dx\right)^{\frac{1}{p}} \\
\text{By General Minkowski} &\le \int \left(\int (f(x-y)g(y))^p dx\right)^{\frac{1}{p}} dy \\
&= \int g(y)||f||_p dy \\
&= ||f||_p ||g||_1
\end{aligned}$$

//

3. Let $(X, \mathcal{F}, \mu)$ be a finite measure space.

a) Prove or disprove: If a sequence $\{f_n\}$ of real valued $\mathcal{F}-$measureable functions on X converge $\mu-$a.e. then $\{f_n\}$ converges in measure.

Proof. Disprove. Consider $(\mathbb{R}, \mathcal{M}, m)$ as a space, and let $f_n = \chi_{[n,n+1]}$, then $f_n \to 0$ a.e. However $\forall n$ we have $m(\{x| \ |f_n(x)| > \frac{1}{2}\}) = 1$, so f_n does not converge in measure to 0. //

b) Prove or disprove: If a sequence $\{f_n\}$ of real valued $\mathcal{F}-$measureable functions on X converge in measure then $\{f_n\}$ converges $\mu-$almost everywhere.

Proof. Disprove. We do this in two steps. First we show that convergence in mean of order p gives convergence in measure. Then we show that convergence in mean of order p does not give convergence $\mu-$a.e. The statement is then false by transitivity. (1) Convergence in L^p gives convergence in measure. Let $\{f_n\} \to f$ in mean of order p. That is $\int |f_n - f|^p \to 0$. Let $\epsilon > 0$. Consider $E_n = \{x| \ |f_n - f| \ge \epsilon\}$. Now consider $\epsilon^p \lim \mu(E_n) = \lim \epsilon^p \mu(E_n) = \lim \int_{E_n} \epsilon^p$. Since $|f_n - f|^p \ge \epsilon^p \Rightarrow \int |f_n - f|^p \ge \int \epsilon^p$ and $\int |f_n - f|^p \to 0$ we have that $\epsilon^p \lim \mu(E_n) = \lim \int_{E_n} \epsilon^p \to 0$, thus $\lim \mu(E_n) \to 0$, and so we converge in measure. However if we consider the space $(\mathbb{R}, \mathcal{M}, m)$, and give a enumeration of the unit interval $I_0 = [0, 1], I_1 = [0, \frac{1}{2}], I_2 = [\frac{1}{2}, 1], I_3 = [0, \frac{1}{3}], I_4 = [\frac{1}{3}, \frac{2}{3}], I_5 = [\frac{2}{3}, 0], \ldots$. Let $f_n = \chi_{I_n}$. Then we have that for $n > \frac{m(m+1)}{2}$ then $\int |f_n| < \frac{1}{m}$ and so $\{f_n\} \to 0$ in mean or order p namely $p = 1$. However $\{f_n\}$ does not converge to 0 a.e., and so this sequence is a sequence which converges in mean of order p, hence in measure, but does not converge $\mu-$a.e. //

c) Prove or disprove: If a sequence $\{f_n\}$ of real valued $\mathcal{F}-$measureable functions on X is Cauchy in $L^1(\mu)$ then $\{f_n\}$ converges in measure

Proof. Prove. Since L^p is complete and the above argument showes that convergence in mean of order p gives convergence in measure, the statement is true. //

4. Let $(X, \mathcal{F}, \mu)$ be a measure space.

a) Prove that if $f \in L^1(\mu)$ then for every $\epsilon > 0$ there is $\delta > 0$ such that $\mu(A) < \delta$ then $\int_A |f| d\mu < \epsilon$.

Proof. Let $\epsilon > 0$ and $f_n = \max(f, n)$. Then we have that $\{f_n\}$ converge monotonically to f. Therefore by monotone convergence we have that $\int f_n \to \int f$. Thus we can find N such that $\int (f - f_N) < \frac{\epsilon}{2}$. Now choose $\delta < \frac{\epsilon}{2 \cdot N}$. Then we have that for any set A such that $\mu(A) < \delta$ then $\int_A f = \int_A f - f_N + f_N = \int_A (f - f_N) + \int_A f_N < \int (f - f_N) + \int_A N \leq \frac{\epsilon}{2} + \frac{\epsilon \cdot N}{2 \cdot N} = \epsilon$ //

b) A sequence $\{f_n\}$ is said to have uniformaly absolutely continuous intergrals if for $\epsilon > 0$ there is $\delta > 0$ such that $\mu(A) < \delta$ then $\int_A |f_n| < \epsilon$ for each n. Suppose $\mu(X) < \infty$ and $\{f_n\} \to f$ has uniformaly absolutely continuous integrals. Then $f_n \to f$ in $L^1(\mu)$.

Proof. We want to show that $\int |f_n - f| \to 0$. Let $\epsilon > 0$. Then we can find $\delta > 0$ such that both $\int_E |f| < \frac{\epsilon}{3}$ and $\int_E |f_n| < \frac{\epsilon}{3}$ for $\mu(E) < \delta$. Now since we are in a finite space, choose A such that $\mu(A) < \delta$ and $f_n \to f$ uniformally on A^c. Now this gives that we can find N such that for $n \geq N$ we have $|f_n - f| < \frac{\epsilon}{3\mu(A^c)}$. Now we have that $\int |f_n - f| \leq \int_A |f_n - f| + \int_{A^c} |f_n - f| \leq \int_A |f_n| + \int_A |f| + \int_{A^c} |f_n - f| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \mu(A^c) \cdot \frac{\epsilon}{3\mu(A^c)} = \epsilon$. Therefore $\{f_n\}$ converges to f in L^1. //

c) Show that (b) imples the Lebesgue Dominated Convergence Theorem

Proof. Lebesgue dominated convergence follows from this as when we have $\{f_n\} \subset L^1(\mu)$ such that $|f| \leq g \in L^1$, then let $\epsilon > 0$. Now let δ be such that $\int_A g < \epsilon$ for $\mu(A) < \delta$. Then we have that $\int_A |f_n| \leq \int_A |g| < \epsilon$ for $\mu(A) < \delta$ as well. This gives that $\{f_n\}$ has absoluetly continuous integrals, which by part (b) gives that $\int |f_n - f| \to 0$ which gives $\int f_n \to \int f$. //

5. Let μ be a signed measure on $\mathcal{F}$ a $\sigma-$algebra. Prove by applying Radon-Nikodym that there is a unique $\mathcal{F}-$measureable function h such that $|h(x)| = 1 \mu-$a.e. and $\mu(A) = \int_A h d|\mu|$ where $|\mu|$ is the total variation measure.

Proof. We have that $\mu^+ << |\mu|$ and $\mu^- << |\mu|$ and in so by Radon-Nikodym each generates a derivative f_1 and f_2 respectively. Further given a Hahn Decomposition (A, B) of X we have that $f_1(x) = 1$ on A and $f_1(x) = 0$ on B, and visa versa for f_2. Now we have that $\mu(E) = \mu^+(E \cap A) - \mu^-(E \cap B) = \int_{E \cap A} f_1 d|\mu| - \int_{E \cap B} f_2 d|\mu| = \int_E (f_1 \cdot \chi_A - f_2 \cdot \chi_B) d|\mu|$. Let $h = f_1 \cdot \chi_A - f_2 \cdot \chi_B$. Now given x we have that either $x \in A$ or $x \in B$. If $x \in A$ then $\chi_B(x) = 0$ and so $|h(x)| = |f_1(x)| = 1$. If $x \in B$ then $\chi_A(x) = 0$ and we have that $|h(x)| = |-f_2(x)| = 1$. //

2002

1. Prove that $\lim\limits_{a\to\infty}\int_1^a \frac{1}{x}\sin(x)dx$ exists but that the function $\frac{1}{x}\sin(x)dx$ is not lebesgue integrable on $[1,\infty)$.

2. Show that if F is non-decreasing function on $[a,b]$, then $F(b)-F(a)\geq \int_a^b F'(t)dt$. Give meaning to the difference between the two quantities when F is right continuous by relating them to the Lebesgue decomposition of F (or of the associated measure), and an example showing that equality does not always hold

3. Prove that the space $L_p(X,\mathcal{F},\mu)$ is complete for $1\leq p<\infty$.

4. Assume the borel set $A\subset[0,1]$ satisfies the following property: there exists $0\leq\tau<1$ such that $m(A\cap I)\leq\tau m(I)$ for all intervals $I\subset[0,1]$. Prove that $m(A)=0$ (Here m is any finite Borel measure on $[0,1]$)

5. Let $(X,\mathcal{S},\mu)$ and $(Y,\mathcal{T},\nu)$ be $\sigma-$finite measure spaces, and let $f:\mathcal{S}\times\mathcal{T}\to\mathbb{R}$ be a $\mathcal{S}\otimes\mathcal{T}-$measurable function. Let $p\geq 1$. Show that if $f(x,y)$ is in $L^p(\mu)$ for every fixed $y\in Y$, then the integral $\int f(x,y)d\nu$ is also in $L^p(\mu)$, even better, prove the generalized Minkowski inequality:

$$(\int_X(\int_Y|f(x,y)d\nu)^p d\mu)^{\frac{1}{p}}\leq\int_Y(\int_X|f(x,y)|^p d\mu)^{\frac{1}{p}}d\nu$$

2002 Solutions

1. Prove that $\lim_{a\to\infty}\int_1^a \frac{1}{x}\sin(x)dx$ exists but that the function $\frac{1}{x}\sin(x)dx$ is not lebesgue integrable on $[1,\infty)$.

Proof. Let $f = \frac{1}{x}\sin(x)$. Consider $\lim_{a\to\infty}\int_1^a fdx$. On each interval $(1,a)$ we have that f is bounded and Riemann-integrable, hence lebesgue integrable and that the two integrals are the same. We have then that $\lim_{a\to\infty}\int_1^a fdx = \lim_{a\to\infty} R\int_1^a fdx = R\int_1^a fdx$. Now f is Riemann integrable on $(1,a)$. So we show that the limit $\lim_{a\to\infty} R\int_1^a \frac{\sin(x)}{x}dx$ converges. We integrate each by parts with $u = \frac{1}{x}$ and $dv = \sin(x)$. Then we have that $du = \frac{-dx}{x^2}$ and $v = -cos(x)dx$ then we have that $\int_1^a \frac{\sin(x)}{x}dx = \frac{-cos(x)}{x}]_1^a - R\int_1^a \frac{cos(x)}{x^2}dx = -cos(1) + \frac{cos(a)}{a} - R\int_1^b \frac{cos(x)}{x^2}dx$. So we evaluate $\lim_{a\to\infty}(-cos(1) + \frac{cos(a)}{a} + R\int_1^a \frac{cos(x)}{x^2}dx) \leq \lim_{a\to\infty}\int_1^a \frac{1}{x^2} + cos(1) < \infty$, thus $\int_1^a f$ exists. However this function is not lebesgue integrable. Consider $\int_1^\infty |f| = \sum_{k=1}^\infty \int_k \pi^(k+1)\pi|f|$. Now on each set $[k\pi,(k+1)\pi]$ we have that $|f|$ is bounded and Riemann-integable, hence it is Lebesgue-integrable with the same integral. However on each such set we have that $\int_{k\pi}^{(k+1)\pi}|f| \geq \frac{\frac{1}{2}\pi}{\frac{(2k+1)\pi}{2}} = \frac{1}{2k+1}$. So we have that $\sum_{k=1}^\infty \int_{k\pi}^{(k+1)\pi}|f| \geq \sum_{k=1}^\infty \frac{1}{2k+1}$. Now since $2k+1 < 3k$ we have $\frac{1}{2k+1} > \frac{1}{3k}$, so we have $\int_1^\infty |f| \geq \sum_{k=1}^\infty \frac{1}{3k} = \infty$. Therefore f is not Lebesgue-integrable. //

2. Show that if F is non-decreasing function on $[a,b]$, then $F(b) - F(a) \geq \int_a^b F'(t)dt$. Give meaning to the difference between the two quantities when F is right continuous by relating them to the Lebesgue decomposition of F (or of the associated measure), and an example showing that equality does not always hold

Proof. Let f be a non-decreasing function on $[a,b]$ and let $g(x) = f'(x)$, and $g_n(x) = n[f(x+\frac{1}{n}) - f(x)]$, then $\{g_n(x)\} \to g$. Redefine f so that $f(x) = f(b)$ for $x \geq b$. Then g_n is measurable and since f is monotone we have $g_n \geq 0$. This gives by Fatou, $\int_a^b g \leq \underline{\lim}\int_a^b g_n = \underline{\lim}\, n\int_a^b (f(x+\frac{1}{n}) - f(x))dm = \underline{\lim}\, n(\int_b^{b+\frac{1}{n}} fdm - \int_a^{a+\frac{1}{n}} fdm) = \underline{\lim}(f(b) - n\int_a^{a+\frac{1}{n}} fdm \leq f(b) - f(a)$.

This equality does not always hold, consider the function $f(x) = 1$ on $[0,1)$ and 3 at $x = 1$. Then f is differentiable almost everywhere and its derivative is $f'(x) = 0$ almost everywhere, thus $\int_a^b f' = 0$ however $f(1) - f(0) = 3 - 1 = 2$ so the inequality is strict. [8] //

3. Prove that the space $L_p(X,\mathcal{F},\mu)$ is complete for $1 \leq p < \infty$.

Proof. It suffices to show that any absolutely summable series in L^p is summable. Give $\sum_{k=1}^\infty f_k$ such that $\{f_k\} \subset L^p$, and $\sum_{k=1}^\infty ||f_k||_p = M < \infty$, then we will show that $\sum_{k=1}^\infty f_k = g \in L^p$ is summable. Consider $h_n = \sum_{k=1}^n |f_k|$. Let h be a measurable function such that $h_n \to h$ μ−a.e. We have that $||h_n||_p \leq$

[8] Lebesgue decomposition was not covered in depth in our class, and so we will omit this section of the proof. If any future generations of UConn math graduate students would like to, please add this section to the proof

$\sum_{k=1}^{n} ||f_k||_p < M$ by Minkowski's inequality. This gives then that $||h_n||_p^p = \int h_n^p \leq M^p$ and so we have $\int h^p \leq M^p$ which gives that h is finite $\mu-$a.e. Where $h(x) < \infty$ we have that $g(x) = \sum_{k=1}^{\infty} f_k(x)$. Since $\sum_{k=1}^{\infty} f_k$ is absolutely summable we have that $\sum_{k=1}^{\infty} |f_k(x)| < \infty$ and so we have that $\sum_{k=1}^{\infty} f_k(x)$ is a real number. Let $g(x) = 0$ where $h(x) = \infty$. Now let $g_n = \sum_{k=1}^{\infty}$, then we have that $|g_n| \leq |h_n| \leq h$ which gives that $|g| \leq h$ and so $g \in L^p$. Now we have that $g_n \nearrow g \quad \mu-$a.e., and since $|g_n - g|^p \leq 2^{p+1}|g|^p$ we have that $\int |g_n - g|^p \to 0$ by dominated convergence, and so $\sum_{k=1}^{\infty} f_k = g$ is summable, whence L^p is complete. //

4. Assume the borel set $A \subset [0,1]$ satisfies the following property: there exists $0 \leq \tau < 1$ such that $m(A \cap I) \leq \tau m(I)$ for all intervals $I \subset [0,1]$. Prove that $m(A) = 0$ (Here m is any finite Borel measure on $[0,1]$)

Proof. Since A is borel, we can find an open set $O = \bigcup I_k$ a disjoint union of open intervals such that $m(O) = m(A) + \epsilon$. This gives that $A = A \cap O = A \cap \bigcup I_k = \bigcup A \cap I_k$, so we have that $m(A) = m(\bigcup A \cap I_k) = \sum m(A \cap I_k) \leq \sum \tau m(I_k) = \tau \sum m(I_k) = \tau m(0) = \tau(m(A) + \epsilon)$. this gives that $m(A) \leq \frac{\tau\epsilon}{1-\tau}$ //

5. Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be $\sigma-$finite measure spaces, and let $f : \mathcal{S} \times \mathcal{T} \to \mathbb{R}$ be a $\mathcal{S} \otimes \mathcal{T}-$measurable function. Let $p \geq 1$. Show that if $f(x,y)$ is in $L^p(\mu)$ for every fixed $y \in Y$, then the integral $\int f(x,y)d\nu$ is also in $L^p(\mu)$, even better, prove the generalized Minkowski inequality:

$$\left(\int_X \left(\int_Y |f(x,y)|d\nu\right)^p d\mu\right)^{\frac{1}{p}} \leq \int_Y \left(\int_X |f(x,y)|^p d\mu\right)^{\frac{1}{p}} d\nu$$

Proof. Consider $h(y) = \int_Y f(x,y)d\nu$. Let q be the conjugate index of p. Then we have that $F : L^q \to \mathbb{R}$ be the functional $F(g) = \int_X g(x)(\int_Y |f(x,y)|d\nu)d\mu$. Then we have that by definition $||F(g)|| = \sup\limits_{||g||_q=1} \int_X g(x)(\int_Y |f(x,y)|d\nu)d\mu$. And so we have that:

$$\begin{aligned}
||F(g)|| &= \sup_{||g||_q=1} \int_X g(x)\left(\int_Y |f(x,y)|d\nu\right)d\mu \\
\text{Tonelli} &= \sup_{||g||_q=1} \int_Y \left(\int_X g(x)|f(x,y)|d\mu\right)d\nu \\
\text{Hölder} &\leq \sup_{||g||_q=1} \int_Y ||g||_q||\ |f(x,y)|\ ||_{p,X}\, d\nu \\
&\leq \int_Y ||\ |f(x,y)|\ ||_{p,X}\, d\nu
\end{aligned}$$

This being the right hand side. We have also that $\int_Y ||\ |f(x,y)|\ ||_{p,X} d\nu$ otherwise the inequality would be trivially solved. This gives then that F is a bounded linear functional on L^q, and so by Riesz-representation we have that $||F|| = ||\int_Y |f(x,y)|d\nu||_{p,X} \Rightarrow ||\int_Y |f(x,y)|d\nu||_{p,X} \leq \int_Y ||\ |f(x,y)|\ ||_{p,X} d\nu$ [9]

//

[9]Note that this technically does not solve the problem in the case when Y is an infinite space, as the right hand side may still be infinite, though the inequality will still hold

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1. a) Let $(X, \mathcal{F}, \mu)$ be a measure space. Define what is meant by saying f is $\mathcal{F}-$measureable.
   b) Give precise definitions of the following modes of convergences:
      i The sequence $\{f_n\}$ converges to f $\mu-$a.e. (AE)
      ii The sequence $\{f_n\}$ converges to f $\mu-$almost uniform (AU)
      iii The sequence $\{f_n\}$ converges to f in measure (M)
      iv The sequence $\{f_n\}$ converges to f in mean of order p.
   c) Prove the two implication AU $\to$ AE and $L^p \to$ M.
   d) Give a diagram for the modes of convergence in the general case where a solid arrow means the first mode always implies the second, and a broken arrow means convergence in the first mode implies a subsequence which converges in the second.

2. a) Give a precise statement of Fatou's Lemma
   b) Suppose $\{f_n\}$ is a sequence of non-negative $\mathcal{F}-$measurable functions converging $\mu-$a.e. to the function f, and suppose $\int f_n d\mu \to \int f d\mu < \infty$. Using only part (a) and the properties of sequences of real numbers, show that for any $A \in \mathcal{F}$ we have $\int_A f_n d\mu \to \int_A f d\mu$.
   c) Indicate where you used the fact that $\int f d\mu < \infty$.

3. a) Let μ be the counting measure on $\wp(\mathbb{N}$. Define g by $g(n) = n^{\frac{-1}{p}}$ where p is a fixed index in $[1, \infty)$. Show that $g \in L^r$ iff $p < r \leq \infty$. Deduce that L^r is not a subset of L^p for $p < r$.
   b) Now let $\mu(n) = \frac{1}{n^2}$, and define $f(n) = n^{\frac{1}{r}}$ where r is a fixed index in $(1, \infty)$. Show that $f \in L^p$ iff $1 \leq p < r$. Deduce that L^p is not a subset of L^r for $p < r$.
   c) For a general X assume $\mu(X) < \infty$ and $1 \leq p < r < \infty$ show $L^r \subset L^p$ and that for $f \in L^r$ we have $||f|||_p \leq ||f||_r \mu(X)^{\frac{1}{p} - \frac{1}{r}}$ (Hint: Note $|f|^p \in L^{\frac{r}{p}}$ and $1 \in L^s$ for all $s \geq 1$).

4. a) Consider two measure spaces $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ where $X = Y = [0, 1]$, and both $\mathcal{F} = \mathcal{G}$ are the $\sigma-$algebra of Borel subsets of $[0, 1]$. If μ is the Lebesgue measure and ν is the counting measure and $D = \{(x, y) | x = y\}$ then D is in the product of the $\sigma-$algebras, however $\int \nu(D_x) d\mu \neq \int \mu(D^y) d\nu$. Why does this not contradict Tonelli's Theorem?
   b) Consider the reals with the Lebesgue measure and the plan with the induced product measure. Let f be the function $f(x, y) = 1$ if $x \geq 0$ and $x \leq y < x + 1$, $f(x, y) = -1$ if $x \geq 0$ and $x + 1 \leq y < x + 2$ and $f(x, y) = 0$ otherwise. Show that $\int [\int f(x, y) dx] dy \neq \int [\int f(x, y) dy] dx$. Why does this not contradict Fubini's Theorem?

5. Let λ and μ be measures on the $\sigma-$algebra $\mathcal{F}$ for a space X. State what it is means for λ to be absolutely continuous with respect to μ ($\lambda << \mu$). Define what is meant by a Radon-Nikodym derivative $d\lambda / d\mu$ Let λ and μ be $\sigma-$finite measures on $(X, \mathcal{F})$, let $\lambda << \mu$, and let $f = d\lambda / d\mu$. If g is a non-negative $\mathcal{F}-$measureable function on X, show that $\int g d\lambda = \int g f d\mu$.

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1.

a) Let $(X, \mathcal{F}, \mu)$ be a measure space. Define what is meant by saying f is $\mathcal{F}-$measureable.

Proof. A function is $\mathcal{F}-$measureable provided that for each $\alpha \in \mathbb{R}$, then $\{x|f(x) > \alpha\} \in \mathcal{F}$. //

b) Give precise definitions of the following modes of convergences:

i The sequence $\{f_n\}$ converges to f $\mu-$a.e. (AE)

ii The sequence $\{f_n\}$ converges to f $\mu-$almost uniform (AU)

iii The sequence $\{f_n\}$ converges to f in measure (M)

iv The sequence $\{f_n\}$ converges to f in mean of order p.

Proof.

$$
\begin{array}{rcl}
AE & \text{is} & \mu(\{x|f_n(x) \text{ does not converge to } f(x)\}) = 0 \\
AU & \text{is} & \forall \eta > 0 \text{ there is a set } A \text{ such that } \mu(A) < \eta \\
 & & \quad \text{and } f_n \to f \text{ uniformly on } A^c. \\
M & \text{is} & \forall \epsilon > 0 \text{ then } \lim_{n\to\infty} \mu(\{x||f_n(x) - f(x)| \geq \epsilon\}) \to 0 \\
L^p & \text{is} & \displaystyle\int |f_n - f|^p \to 0
\end{array}
$$

//

c) Prove the two implication AU $\to$ AE and $L^p \to$ M.

Proof. (AU $\to$ AE). Let f_n converge to f AU. Then for $k \in \mathbb{N}$ we can find A_k such that $\mu(A_k) < \frac{1}{k}$ and $f_n \to f$ uniformly on A_k^c. Let $A = \bigcap A_k$. Then we have that for $x \in A^c = \bigcup A_k^c$ then $f_n(x) \to f(x)$, and since $\mu(A) < \frac{1}{k} \forall k$ we have that $\mu(A) = 0$ and so $f_n \to f$ AE.

($L^p \to$ M) Let $\int |f_n - f|^p \to 0$. Then for any $\epsilon > 0$ let $E_n = \{x|\ |f_n(x) - f(x)| \geq \epsilon\}$. Then we have that $\int_{E_n} |f_n - f|^p \geq \epsilon^p \mu(E_n)$. But since $\int |f_n - f|^p \to 0$ as $n \to \infty$, we have that $\epsilon^p \mu(E_n) \to 0$ which gives that $\mu(E_n) \to 0$ as $n \to \infty$. //

d) Give a diagram for the modes of convergence in the general case where a solid arrow means the first mode always implies the second, and a broken arrow means convergence in the first mode implies a subsequence which converges in the second.

2.

a) Give a precise statement of Fatou's Lemma

Proof. Let $\{f_n\}$ be a sequence of non-negative functions converging almost everywhere. Then for any measurable set A we have that $\int_A \underline{\lim} f_n \leq \underline{\lim} \int_A f_n$. //

b) Suppose $\{f_n\}$ is a sequence of non-negative $\mathcal{F}-$measurable functions converging $\mu-$a.e. to the function f, and suppose $\int f_n d\mu \to \int f d\mu < \infty$. Using only part (a) and the properties of sequences of real numbers, show that for any $A \in \mathcal{F}$ we have $\int_A f_n d\mu \to \int_A f d\mu$.

Proof. Let $\{f_n\}$ be a sequence of non-negative $\mathcal{F}-$measurable functions converging almost eveywhere to f where $\int f d\mu < \infty$. Let E be any measurable set. By Fatou's lemma we have $\int_E f \leq \underline{\lim} \int_E f_n$. Now we consider $\overline{\lim} \int_E f_n = \overline{\lim}(\int f_n - \int_{E^c} f_n) \leq \overline{\lim}(\int f_n) + \overline{\lim}(-\int_{E^c} f_n) = \int f - \underline{\lim}(\int_{E^c} f_n)$. Again Fatou gives us $\int_{E^c} f \leq \underline{\lim}(\int_{E^c} f_n) \Rightarrow -\int_{E^c} f \geq -\underline{\lim}(\int_{E^c} f_n)$, thus we have $\int f - \underline{\lim}(\int_{E^c} f_n) \leq \int f - \int_{E^c} f = \int_E f$. Thus $\overline{\lim} \int_E f_n \leq \int_E f \Rightarrow \int_E f_n \to \int_E f$. //

c) Indicate where you used the fact that $\int f d\mu < \infty$.

Proof. We used this when we computed the expression $\int f - \underline{\lim}(\int_{E^c} f_n)$. If $\int f = \infty$ then this expression makes no sense. //

3.

a) Let μ be the counting measure on $\wp(\mathbb{N}$. Define g by $g(n) = n^{\frac{-1}{p}}$ where p is a fixed index in $[1, \infty)$. Show that $g \in L^r$ iff $p < r \leq \infty$. Deduce that L^r is not a subset of L^p for $p < r$.

Proof. We compute $\int g^r d\mu = \sum_{n=1}^{\infty} \int_n g^r d\mu = \sum_{n=1}^{\infty} g^r(n) = \sum_{n=1}^{\infty} n^{\frac{-r}{p}}$ which will converge exactly when $\frac{r}{p} > 1$ or $r > p$. //

b) Now let $\mu(n) = \frac{1}{n^2}$, and define $f(n) = n^{\frac{1}{r}}$ where r is a fixed index in $(1, \infty)$. Show that $f \in L^p$ iff $1 \leq p < r$. Deduce that L^p is not a subset of L^r for $p < r$.

Proof. Now we compute $\int f d\mu = \sum_{n=1}^{\infty} \int_n f^p d\mu = \sum_{n=1}^{\infty} \frac{f^p(n)}{n^2} = \sum_{n=1}^{\infty} \frac{n^{\frac{p}{r}}}{n^2} = \sum_{n=1}^{\infty} n^{\frac{p}{r}-2}$ which converges exactly when $\frac{p}{r} - 2 < -1$ or $p < r$. //

c) For a general X assume $\mu(X) < \infty$ and $1 \leq p < r < \infty$ show $L^r \subset L^p$ and that for $f \in L^r$ we have $||f|||_p \leq ||f||_r \mu(X)^{\frac{1}{p}-\frac{1}{r}}$ (Hint: Note $|f|^p \in L^{\frac{r}{p}}$ and $1 \in L^s$ for all $s \geq 1$

Proof. Let $1 \leq p < r < \infty$. Then we have that $f^p \in L^{\frac{r}{p}}$. Let q be the congugate index for $\frac{r}{p}$. Since our space is finite we have that $1 \in L^q$ for any q. Thus we have that $||f||_p^p = \int |f|^p \leq ||f^p||_{\frac{r}{p}} ||1||_q = \left(\int (|f|^p)^{\frac{r}{p}}\right)^{\frac{p}{r}} \left(\int 1\right)^{\frac{1}{q}} = ||f||_r^p \mu(X)^{\frac{1}{q}}$. by Hölders inequality. Since $\frac{p}{r} + \frac{1}{q} = 1$ we have that $\frac{1}{q} = \frac{r-p}{r} = 1 - \frac{p}{r}$. Now taking $p-$th roots of both sides we have $||f||_p \leq ||f||_r \mu(X)^{\frac{1}{p}(1-\frac{p}{r})} = ||f||_r \mu(X)^{\frac{1}{p}-\frac{1}{r}}$. //

4.

a) Consider two measure spaces $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ where $X = Y = [0, 1]$, and both $\mathcal{F} = \mathcal{G}$ are the $\sigma-$algebra of Borel subsets of $[0, 1]$. If μ is the Lebesgue measure and ν is the counting measure and $D = \{(x, y) | x = y\}$ then D is in the product of the $\sigma-$algebras, however $\int \nu(D_x) d\mu \neq \int \mu(D^y) d\nu$. Why does this not contradict Tonelli's Theorem?

Proof. D is in the product $\sigma-$algebra. Let $(x,y) \in X \times Y$ such that $x \neq y$, and choose rational $x < q < y$. Then we have that either $(x,y) \in (q,1) \times (0,q)$ or $(0,q) \times (q,1)$. And so we can $D^c = \bigcup_{q \in \mathbb{Q}}(q,1)(\times(0,q) \cup (0,q) \times (q,1)) \in \mathcal{F} \times \mathcal{G}$, and so D is also. Now $D_x = \{x\}$ and $D^y = \{y\}$ which gives that $\int \nu(D_x)d\mu = \int \nu(\{x\})d\mu = \int 1 d\mu = 1$ but $\int \mu(D^y)d\nu = \int \mu(D^y)d\nu = \int 0 d\nu = 0$ so $\int \nu(D_x)d\mu \neq \int \mu(D^y)d\nu$. This does not contradict Tonelli's theorem since $(Y, \mathcal{G}, \nu)$ is not a $\sigma-$finite space. //

b) Consider the reals with the Lebesgue measure and the plan with the induced product measure. Let f be the function $f(x,y) = 1$ if $x \geq 0$ and $x \leq y < x+1$, $f(x,y) = -1$ if $x \geq 0$ and $x+1 \leq y < x+2$ and $f(x,y) = 0$ otherwise. Show that $\int[\int f(x,y)dx]dy \neq \int[\int f(x,y)dy]dx$. Why does this not contradict Fubini's Theorem?

Proof. Let f be the function described. For fixed x we have that $\int f(x,y)dy = \int_{-\infty}^{x} 0dy + \int_{x}^{x+1} 1dy + \int_{x+1}^{x+2} -1dy + \int_{x+2}^{\infty} 0dy = x+1-x-x-2+x+1 = 0$, and hence $\int[\int f(x,y)dy]dx = 0$. We have also that $\int f(x,y)dx = 0$ for $y \leq 0$. For $y \geq 2$ we have $\int f(x,y)dx = \int_{y-1}^{y} dx - \int_{y-2}^{y-1} dx = y-y+1-y+1+y-2 = 0$. However for $1 < y < 2$ we have $\int f(x,y)dx = \int_{y-1}^{y} dx - \int_{0}^{y-1} dx = y-y+1-y+2 = 2-y$, and for $0 < y < 1$ we have $\int f(x,y)dx = \int_0^y dx = y$. This gives then that $\int(\int f(x,y)dx)dy) = \int_0^1 ydy + \int_1^2 (2-y)dy = \frac{1}{2} - 4 - 2 - 2 + \frac{1}{2} = 1$ and so $\int[\int f(x,y)dx]dy \neq \int[\int f(x,y)dy]dx$ This does not contradict Fubini's theorem since our function f is not lebesgue integrable $\int\int |f| = \infty$. //

5. Let λ and μ be measures on the $\sigma-$algebra $\mathcal{F}$ for a space X. State what it is means for λ to be absolutely continuous with respect to μ ($\lambda << \mu$). Define what is meant by a Radon-Nikodym derivative $[d\lambda/d\mu]$ Let λ and μ be $\sigma-$finite measures on $(X, \mathcal{F})$, let $\lambda << \mu$, and let $f = d\lambda/d\mu$. If g is a non-negative $\mathcal{F}-$measureable function on X, show that $\int g d\lambda = \int g f d\mu$.

Proof. We mean by $\lambda << \mu$ that if $\mu(A) = 0$ then $\lambda(A) = 0$. A Radon-Nikodym derivative $[d\lambda/d\mu]$ is a unique (almost everywhere) non-negative measurable function such that $\lambda(E) = \int_E [d\lambda/d\mu]d\mu$. We begin by first showing the result for the case when g is simple. Let $g = \sum_{k=1}^n a_k \chi_{A_k}$ be a simple function. Then we have that $\int g d\lambda = \int \sum_{k=1}^n a_k \chi_{A_k} d\lambda = \sum_{k=1}^n a_k \lambda(A_k) = \sum_{k=1}^n a_k \int_{A_k} f d\mu = \int(\sum_{k=1}^n a_k \chi_{A_k}) \cdot f d\mu = \int g \cdot f d\mu$. Now let g be a non-negative $\mathcal{F}-$measurable function. We can find a sequence of simple functions $\{\varphi_n\}$ such that $\varphi_n \nearrow g \mu-$a.e, whence $\lambda-$a.e. Now by monotone convergence we have that $\int \varphi_n d\lambda \to \int g d\lambda$. However by our previous results we have also that $\int \varphi_n d\lambda = \int \varphi_n \cdot f d\mu$. Since $\varphi_n \nearrow g \mu-$a.e. then $\varphi_n \cdot f \nearrow g \cdot f \mu-$a.e. and so again by monotone convergence we obtain $\int \varphi_n d\lambda = \int \varphi_n \cdot f d\mu \to \int g \cdot f d\mu$. Uniqueness of limit gives that $\int g d\lambda = \int g \cdot f d\mu$. //

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Note All spaces are assumed to be $\sigma-$finite.

1. Prove or disprove:
   - i Every Riemann-integrable function on $[0,\infty)$ is Lebesgue-integrable.
   - ii Every positive Riemann-integrable function on $[0,\infty)$ is Lebesgue-integrable.
   - iii Every Lebesgue-integrable function on $[0,\infty)$ is Riemann-integrable
2. Suppose $(X,\mathcal{F},\mu)$ is a finite measure space. Prove or disprove:
   - i Every sequence of $\mathcal{F}-$measurable functions that converges in the $L^1(\mu)-$norm converge a.e. (μ)
   - ii Every sequence of $\mathcal{F}-$measurable functions that converges a.e. (μ) converges in measure μ.
   - iii Every sequence of $\mathcal{F}-$measureable functions that converge in the $L^1(\mu)-$norm converges in measure
3.
   - i Prove that if $f \in L^1(\mu)$ then $\lambda(E) = \int_E f d\mu$ defines a $\mathbb{C}-$measure on $(X,\mathcal{F})$.
   - ii What is the Radon-Nikodym derivative of λ with respect to mu?
   - iii Prove from first principles (without invoking any "big-name" theorems) that if $\lambda(E) = 0$ for all $E \in \mathcal{F}$ then $f = 0$ a.e. (μ)
4.
   - i Prove the Generalized Minkowski inequality: If $(X,\mathcal{F},\mu)$ and $(Y,\mathcal{G},\lambda)$ are measure spaces, f a $(\mathcal{F}\times\mathcal{G})-$measurable function on $X\times Y$ and $p \in [1,\infty)$, then
   $$(\int_X (\int_Y |f(x,y)| d\lambda)^p d\mu)^{\frac{1}{p}} \leq \int_Y (\int_X |f(x,y)|^p d\mu)^{\frac{1}{p}} d\lambda$$
   - ii Explain how the inequality above generalizes the Minkowski inequality
5. Suppose that f is a $\mathbb{C}-$valued measurable function on a measure space (X,μ).
   - i Prove that if $||f||_r < \infty$ for some $r < \infty$ then $||f||_p \to ||f||_\infty$ as $p \to \infty$.
   - ii Prove that if $\mu(X) = 1$[10] and $||f||_r < \infty$ for some $r > 0$ then
   $$\lim_{p\to 0} ||f||_p = e^{\int_x \log|f| d\mu}$$
   .
6.
   - i Supppose $\{f_n\}$ is a sequence of functions in L^p with $p \in (1,\infty)$ which converge a.e. (μ) to $f \in L^p$. Prove that if $||f_n||_p \leq 1$ for all n, then for all $g \in L^q$ $(\frac{1}{p}+\frac{1}{q}=1)$ then
   $$\int_X f_n \, g d\mu \to \int f g d\mu \text{ as } n \to \infty$$
   - ii Does your proof (of the statement in i) extend to $p = 1$ and $q = \infty$? If it does, then show how, if it does not , then show where the proof fails, and produce a counterexample.

[10]This was added in during the prelim

2000 Solutions

Note All spaces are assumed to be $\sigma-$finite.

1. Prove or disprove:

i Every Riemann-integrable function on $[0,\infty)$ is Lebesgue-integrable.

Proof. Disprove. Consider $f(x)=\frac{\sin(x)}{x}$ on $[1,\infty)$ and 0 on $[0,1]$. This is Riemann-integrable provided the limit $\lim\limits_{b\to\infty} R\int_1^b \frac{\sin(x)}{x}dx$ converges. We integrate each by parts with $u=\frac{1}{x}$ and $dv=\sin(x)$. Then we have that $du=\frac{-dx}{x^2}$ and $v=-cos(x)dx$ then we have that $\int_1^b \frac{\sin(x)}{x}dx = \frac{-cos(x)}{x}]_1^b - R\int_1^b \frac{cos(x)}{x^2}dx = -cos(1)+\frac{cos(b)}{b} - R\int_1^b \frac{cos(x)}{x^2}dx$. So we evaluate $\lim\limits_{b\to\infty}(-cos(1)+\frac{cos(b)}{b}+R\int_1^b \frac{cos(x)}{x^2}dx) \le \lim\limits_{b\to\infty}\int_1^b \frac{1}{x^2}+cos(1)<\infty$, thus f is Riemann-integrable. Conversly to compute $\int_1^\infty |f| = \sum_{k=1}^\infty \int_k \pi^(k+1)\pi|f|$. Now on each set $[k\pi,(k+1)\pi]$ we have that $|f|$ is bounded and Riemann-integable, hence it is Lebesgue-integrable with the same integral. However on each such set we have that $\int_{k\pi}^{(k+1)\pi}|f| \ge \frac{\frac{1}{2}\pi}{\frac{(2k+1)\pi}{2}} = \frac{1}{2k+1}$. So we have that $\sum_{k=1}^\infty \int_{k\pi}^{(k+1)\pi}|f| \ge \sum_{k=1}^\infty \frac{1}{2k+1}$. Now since $2k+1<3k$ we have $\frac{1}{2k+1}>\frac{1}{3k}$, so we have $\int_1^\infty |f| \ge \sum_{k=1}^\infty \frac{1}{3k}=\infty$. Therefore f is not Lebesgue-integrable. //

ii Every positive Riemann-integrable function on $[0,\infty)$ is Lebesgue-integrable.

Proof. We prove this in two steps. First we show that if f is Riemann-integrable on a set $[a,b]$ then it is lebesgue integrable. Then we will extend this result to be true on $(0,\infty)$. Let f be a Riemann-integrable function on $[a,b]$ then we will show that this is lebesgue integrable and the two integrals are equal. Let h be any bounded measurable function $h\le f$. Then, $R\underline{\int}_a^b h \le \sup\limits_{\varphi\le h}\int_a^b \varphi \le \inf\limits_{\varphi\ge h}\int_a^b \varphi \le R\overline{\int}_a^b h$ where φ are simple functions. These are true since simple functions are step functions. Then we have that $\forall h$ we have that $\int_a^b h \le R\overline{\int}_a^b h \le R\overline{\int}_a^b f = R\int_a^b f<\infty$ thus we have that $\int_a^b f = \sup h \le f\int_a^b h \le R\int_a^b f<\infty$. Now we have that $R\int_a^b f = R\underline{\int}_a^b f = \sup\psi \le fR\int_a^b \psi$ where ψ are taken to be step functions. Since these are all bounded Riemann-integrable functions on a finite interval we have that they are lebesgue integrable and the two integrals are equal, thus $\sup\psi\le fR\int_a^b\psi = \sup\psi\le f\int_a^b \psi$ and since every step function is also bounded measurable we have that $R\int_a^b f = \sup\psi \le f\int_a^b\psi \le \sup\limits_{h\le f}\int_a^b h = \int_a^b f$ by definition, so $R\int_a^b f = \int_a^b f$. Now we will extend this to $(0,\infty)$. Let f be Riemann-integrable on $(0,\infty)$ and consider $f_n = f\chi_{(0,n)}$ then $f_n \nearrow f$ and are non-negative, thus $\lim\limits_{n\to\infty} R\int_0^n f = \lim\limits_{n\to\infty}\int_0^n f$ by the previous argument. And so we have $\lim\limits_{n\to\infty}\int_0^n f = \lim\limits_{n\to\infty}\int f_n \to \int_0^\infty f$ by monotone convergence. Likewise we have $\lim\limits_{n\to\infty} R\int_0^n f = R\int_0^\infty f<\infty$ by definition, and so by uniqueness of limit we have that $\int_0^\infty f = R\int_0^\infty f < \infty$ so f is lebesgue integrable. //

iii Every Lebesgue-integrable function on $[0, \infty)$ is Riemann-integrable

Proof. Prove. Consider the function $f(x) = \chi_{\mathbb{Q}}$ on $[0, \infty)$ is a function which is not Riemann-integrable as it is discontinuous at every point in $(1, \infty)$. However this is Lebesgue-integrable and its Lebesgue-integral is 0 as it is only non-zero on a set of zero measure. //

2. Suppose $(X, \mathcal{F}, \mu)$ is a finite measure space. Prove or disprove:

i Every sequence of $\mathcal{F}-$measurable functions that converges in the $L^1(\mu)-$norm converge a.e. (μ)

Proof. Disprove. Consider $(X, \mathcal{M}, [0,1])$ and $I_0 = [0,1], I_1 = [0, \frac{1}{2}], I_2 = [\frac{1}{2}, 1], I_3 = [0, \frac{1}{3}], I_4 = [\frac{1}{3}, \frac{2}{3}], I_5 = [\frac{2}{3}, 0], \ldots$. Let $f_n = \chi_{I_n}$. Then we have that for $n > \frac{m(m+1)}{2}$ then $\int |f_n| < \frac{1}{m}$ and so $\{f_n\} \to 0$ in mean or order p namely $p = 1$. However $\{f_n\}$ does not converge to 0 a.e., and so this sequence is a sequence which converges in mean of order p, hence in measure, but does not converge $\mu-$a.e. //

ii Every sequence of $\mathcal{F}-$measurable functions that converges a.e. (μ) converges in measure μ.

Proof. Prove. This is true by two steps. First we show that for finite spaces convergence $\mu-$a.e. gives convergence almost uniform. Then we show that convergence almost uniform gives convergence in measure. Let $f_n \to f \mu-$a.e. Then given $\epsilon, \delta > 0$. Let A_1 be the set of $x \in X$ such that $\{f_n(x)\}$ does not converge to f. Then $\mu(A_1) = 0$. Now let $G_k = \{x \notin A_1 | \, |f_k(x) - f(x)| \geq \epsilon\}$. Let $E_n = \bigcup_{k \geq n} G_k$. Then $X \supset E_1 \supset E_2 \supset \ldots$. Further $\bigcap_n E_n = \emptyset$. Since $\mu(X) < \infty$ we have that $\lim \mu(E_n) = \mu(\bigcap E_n) = 0$. Thus we can find N large enough such that $\mu(E_N) < \delta$ and $\forall N \geq n$ we have $|f_n(x) - f(x)| < \epsilon$ by definition. $\forall k \in \mathbb{N}$ let $\epsilon_k = \frac{1}{k}$ and $\delta_k = 2^{-k}\eta$. Use the previous argument to choose sets A_k and numbers N_k such that $\mu(A_k) < \delta_k$ and $\forall n \geq N_k$ we have $|f_n(x) - f(x)| < \epsilon_k$ on A_k^c. Let $A = \bigcup A_k$. Then $\mu(A) \leq \sum 2^{-k}\eta = \eta$. Let $\epsilon > 0$. Choose k such that $\frac{1}{k} < \epsilon$, then $\forall x \in A^c \subset A_k^c$ and $n \geq N_k$ we have $|f_n(x) - f(x)| \leq \frac{1}{k} < \epsilon$. Now given this we show that $f_n \to f$ in measure. We have that $\forall \epsilon > 0$ there is an N such that $\forall n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$ on A^c, thus $\mu(\{x| \, |f_n(x) - f(x)| \geq \epsilon\}) \leq \mu(A) < \eta$ for $n \geq N$. Therefore since η was arbitrary, we have $\{f_n\}$ converges in meausure to f. //

iii Every sequence of $\mathcal{F}-$measureable functions that converge in the $L^1(\mu)-$norm converges in measure

Proof. Prove. Let $\{f_n\}$ be a sequence which converges in mean of order p. Given $\epsilon > 0$ let $E_n = \{x| \, |f_n(x) - f(x)| \geq \epsilon\}$ then we can write $\mu(E_n) = \int_{E_n} 1 d\mu$. We have also that $|f_n(x) - f(x)| \geq \epsilon \Rightarrow |f_n(x) - f(x)|^p \geq \epsilon^p$. Then $\epsilon^p \mu(E_n) = \int_{E_n} \epsilon^p \leq \int_{E_n} |f_n(x) - f(x)|^p \leq \int |f_n(x) - f(x)|^p \to 0$ as

$n \to \infty$, therefore we have that $\epsilon^p \mu(E_n) \to 0$ as $n \to \infty$ which gives that $\mu(E_n) \to 0$ as $n \to \infty$. //

3.

i Prove that if $f \in L^1(\mu)$ then $\lambda(E) = \int_E f d\mu$ defines a $\mathbb{C}-$measure on $(X, \mathcal{F})$.

Proof. We have that $\lambda(\emptyset) = \int_\emptyset f d\mu = 0$ as μ is a measure. Likewise let $\{E_k\}$ be a countable disjoint family of sets. Then $\lambda \bigcup E_k = \int_{\bigcup E_k} f d\mu = \int \sum_{k=1}^\infty f\chi_{E_k} d\mu$. Now let $g_n = \sum_{k=1}^n f\chi_{E_k}$. Then we have that $g_n \to f \mu-$a.e. and $|g_n| \leq f \in L^1$ so we can write $\int \sum_{k=1}^\infty f\chi_{E_k} d\mu = \int \lim g_n d\mu = \lim \int g_n d\mu = \lim \int \sum_{k=1}^n f\chi_{E_k} d\mu = \lim \sum_{k=1}^n \int f\chi_{E_k} d\mu = \sum_{k=1}^\infty \int_{E_k} f d\mu = \sum_{k=1}^\infty \lambda(E_k)$. This is finite and absolutely convergent as $\int |f| d\mu < \infty$, and hence λ is a complex measure. //

ii What is the Radon-Nikodym derivative of λ with respect to mu?

Proof. The Radon-Nikodym derivative of λ with repsect to μ is only defined when $\lambda << \mu$, or when $\mu(E) = 0 \Rightarrow \lambda(E) = 0$, which is true in our case. In this case, the Radon-Nikodym derivative $[\frac{d\lambda}{d\mu}]$ is the unqiue (almost everywhere) non-negative measurable function such that $\lambda(E) = \int_E [\frac{d\lambda}{d\mu}] d\mu$. //

iii Prove from first principles (without invoking any "big-name" theorems) that if $\lambda(E) = 0$ for all $E \in \mathcal{F}$ then $f = 0$ a.e. (μ)

Proof. Let $f \neq 0$ on a set E such that $\mu(E) > 0$. Without loss of generality assume $f > 0$ on E (either $f > 0$ or $f < 0$ on a smaller set of positive measure). Then let $E_0 = \{x \in E | f(x) \geq 1\}$, and $E_n = \{x \in E | \frac{1}{n} > f(x) \geq \frac{1}{n+1}\}$. Then if $\mu(E_n) = 0$ for any $n \in \mathbb{N} \cup \{0\}$we have that $E = \bigcup E_n$ so $\mu(E) = \sum \mu(E_n) = 0$ which is a contradiction. Let n be such that $\mu(E_n) > 0$. If $n = 0$, then $\int_{E_n} f d\mu \geq 1\mu(E_n) > 0$ and otherwise $\int_{E_n} f d\mu \geq \frac{\mu(E_n)}{n} > 0$. //

4.

i Prove the Generalized Minkowski inequality: If $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \lambda)$ are measure spaces, f a $(\mathcal{F} \times \mathcal{G})-$measurable function on $X \times Y$ and $p \in [1, \infty)$, then

$$(\int_X (\int_Y |f(x,y)| d\lambda)^p d\mu)^{\frac{1}{p}} \leq \int_Y (\int_X |f(x,y)|^p d\mu)^{\frac{1}{p}} d\lambda$$

Proof. Consider $h(y) = \int_Y f(x,y) d\lambda$. Let q be the conjugate index of p. Then we have that $F : L^q \to \mathbb{R}$ be the functional $F(g) = \int_X g(x) (\int_Y |f(x,y)| d\lambda) d\mu$. Then we have that by definition $||F(g)|| = \sup_{||g||_q = 1} \int_X g(x) (\int_Y |f(x,y)| d\lambda) d\mu$.

And so we have that:

$$\begin{aligned} ||F(g)|| &= \sup_{||g||_q=1} \int_X g(x)(\int_Y |f(x,y)|d\lambda)d\mu \\ \text{Tonelli} &= \sup_{||g||_q=1} \int_Y (\int_X g(x)|f(x,y)|d\mu)d\lambda \\ \text{Hölder} &\leq \sup_{||g||_q=1} \int_Y ||g||_q|| \; |f(x,y)| \; ||_{p,X} d\lambda \\ &\leq \int_Y || \; |f(x,y)| \; ||_{p,X} d\lambda \end{aligned}$$

This being the right hand side. We have also that $\int_Y || \; |f(x,y)| \; ||_{p,X} d\lambda$ otherwise the inequality would be trivially solved. This gives then that F is a bounded linear functional on L^q, and so by Riesz-representation we have that $||F|| = || \int_Y |f(x,y)|d\lambda||_{p,X} \Rightarrow$

$$\Big(\int_X (\int_Y |f(x,y)|d\lambda)^p d\mu)^{\frac{1}{p}} \leq \int_Y (\int_X |f(x,y)|^p d\mu)^{\frac{1}{p}} d\lambda$$

//

ii Explain how the inequality above generalizes the usual Minkowski inequality

Proof. Let $Y = \mathbb{N}$, $\mathcal{G} = \wp(N)$ and λ be the counting measure. Then we have that $\big(\int_X (\int_Y |f(x,y)d\lambda)^p d\mu)^{\frac{1}{p}} \leq \int_Y (\int_X |f(x,y)|^p d\mu)^{\frac{1}{p}} d\lambda$ gives $(\int_X (\sum f(x,k))^p d\mu)^{\frac{1}{p}} \leq \sum (\int_X |f(x,k)|^p d\mu)^{\frac{1}{p}}$. That is to say $||\sum f_k||_{p,X} \leq \sum ||f_k||_{p,X}$ which is a generalization of Minkowski's inequality to infinte sums. //

5. Suppose that f is a $\mathbb{C}-$valued measurable function on a measure space (X,μ).

i Prove that if $||f||_r < \infty$ for some $r < \infty$ then $||f||_p \to ||f||_\infty$ as $p \to \infty$.

Proof. Let $f \in L^r$. We have the following cases. If $f \in L^\infty$, let $M = ||f||_\infty$. Then we have that $\forall \epsilon > 0$ we have that for $B_\epsilon = \{x| \; |f(x)| > M - \epsilon\}$, then we have that $(\int |f|^p)^{\frac{1}{p}} > (\int_{B_\epsilon} |f|^p)^{\frac{1}{p}} > \mu(B_\epsilon)^{\frac{1}{p}}(M-\epsilon)$. We have that $\mu(B_\epsilon) < \infty$ since $(\int |f|^r)^{\frac{1}{r}} > \mu(B_\epsilon)^{\frac{1}{r}}(M-\epsilon)$. Therefore we have that $\varliminf_{p\to\infty} ||f||_p \geq ||f||_\infty$. Now $||f||_p^p = \int |f|^p d\mu = \int |f|^r |f|^{p-r}$. We have then that $f^r \in L^\infty$. And since $f \in L^\infty$ then $f^{p-r} \in L^\infty$, thus by Hölder $\int |f|^r |f|^{p-r} d\mu \leq ||f^r||_1 ||f^{p-r}||_\infty = ||f||_r^r ||f||_\infty^{p-r}$. Taking $p-$th roots we have $||f||_p \leq ||f||_r^{\frac{r}{p}} ||f||_\infty^{\frac{p-r}{p}}$, and taking limits $\varlimsup_{p\to\infty} ||f||_p \leq ||f||_\infty$ and so $\lim_{p\to\infty} ||f||_p = ||f||_\infty$. Given that $f \notin L^\infty$ then trivially $\varlimsup_{p\to\infty} ||f||_p \leq ||f||_\infty$. Then we have that $\forall M > 0$ we have that for $B_M = \{x| \; |f(x)| > M\}$, then $(\int |f|^p)^{\frac{1}{p}} > (\int_{B_M} |f|^p)^{\frac{1}{p}} > \mu(B_M)^{\frac{1}{p}} M$. We have that $\mu(B_M) < \infty$ since $(\int |f|^r)^{\frac{1}{r}} > \mu(B_M)^{\frac{1}{r}}(M)$. Therefore we have that $\varliminf_{p\to\infty} ||f||_p \geq ||f||_\infty$. //

ii Prove that if $\mu(X) = 1$ and $||f||_r < \infty$ for some $r > 0$ then

$$\lim_{p\to 0} ||f||_p = e^{\int_x \log|f| d\mu}$$

.

Proof. Consider $\lim_{p\to 0^+} \log ||f||_p = \lim_{p\to 0^+} \log((\int |f|^p)^{\frac{1}{p}}) = \lim_{p\to 0^+} \frac{1}{p}\log(\int |f|^p)$ Now we have an indeterminate form, and so by L'Hopitals rule we have $\lim_{p\to 0^+} \log ||f||_p = \lim_{p\to 0^+} \frac{\frac{d}{dp}\int |f|^p}{\int |f|^p}$. In the numerator since we are not differentiating with respect to the variable of integration, we can pass the derivative under the integral sign to obtain $\lim_{p\to 0^+} \frac{\int \frac{d}{dp}|f|^p}{\int |f|^p} = \lim_{p\to 0^+} \frac{\int |f|^p \log(|f|)}{\int |f|^p}$ Now taking the limit we have $\lim_{p\to 0^+} \log ||f||_p = \frac{\int \log(|f|)}{\int 1} = \int \log(|f|)$ by hypothesis. Exponentiating both sides yields the result. //

6.

i Supppose $\{f_n\}$ is a sequence of functions in L^p with $p \in (1,\infty)$ which converge a.e. (μ) to $f \in L^p$. Prove that if $||f_n||_p \leq 1$ for all n, then for all $g \in L^q$ $(\frac{1}{p} + \frac{1}{q} = 1)$ then

$$\int_X f_n \, g d\mu \to \int f g d\mu \text{ as } n \to \infty$$

Proof. Let $\{f_n\} \to f$ in L^p. We will first prove our theorem for the case when g is simple, and then extend to the more general settings. Let $g = \sum_{i=1}^k a_i \chi_{A_i}$. Since our space is $\sigma-$finite we can make g finitely supported. Now consider $\int f_n g - \int fg = \int_{\bigcup A_i} |f_n - f||g| \leq ||g||_\infty \int_{\bigcup A_i} |f_n - f|$. Let $\epsilon > 0$. then by Egoroff, we can find a subset $A \subset \bigcup A_i$ such that $\mu(A) < \frac{\epsilon}{2||g||_\infty(1+||f||_p)}$ and $f_n \to f$ uniformly on $E = \bigcup A_i \setminus A$. We have also that there is N such that $\forall n \geq N$ then $|f_n(x) - f(x)| < \frac{\epsilon}{2||g||_\infty \mu(E)}$. Then we have that $||g||_\infty \int_{\bigcup A_i} |f_n - f||g| = \int_E |f_n - f||g|d\mu + \int_A |f_n - f||g|d\mu < \frac{\epsilon}{2} + ||g||_\infty \int |f_n - f| \leq \frac{\epsilon}{2} + ||g||_\infty ||f_n - f||_p \mu(A)$ by Hölder. We have then by hypothesis that $\frac{\epsilon}{2} + ||g||_\infty ||f_n - f||_p \mu(A) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon$ for $n \geq N$. So we are done for g simple. Then for the general case $\epsilon > 0$ choose φ a simple function such that $||g - \varphi||_q < \frac{\epsilon}{2(1+||f||_p)}$ Then we have that $\int |f_n - f||g| = \int |f_n - f||g - \varphi| + \varphi| = \int |f_n - f||g - \varphi| + \int |f_n - f||\varphi|$. We can make $\int |f_n - f||\varphi| < \frac{\epsilon}{2}$ by the previous argument. Likewise by Hölder we have $\int |f_n - f||g - \varphi| \leq ||f_n - f||_p ||g - \varphi||_q < \frac{\epsilon}{2}$ and so we are done. //

ii Does your proof (of the statement in i) extend to $p = 1$ and $q = \infty$? If it does, then show how, if it does not , then show where the proof fails, and produce a counterexample.

Proof. This does not hold. Consider $(\mathbb{R}, \mathcal{M})$ and take $g = 1$. Then $g \in L^\infty$. However given $f_n = n\chi_{(0,\frac{1}{n})}$. Then $\int f_n = 1$ for all n however $f_n \to 0$ a.e. and so $\int f = 0$. //

1999

1. Prove that if $\{f_n\}$ are non-negative measurable functions that converge a.e. on $\mathbb{R}$ and $\int_{\mathbb{R}} f_n \to \int_{\mathbb{R}} f < \infty$. Then show:

a) $\int_E f = \lim \int_E f_n$ for any measurable sets
b) Show that this needs not be the case if $\int_{\mathbb{R}} f = \infty$

2. $f = x^{\frac{-1}{2}} \chi_{(0,1)}$. Then

a) Show f is integrable on $\mathbb{R}$
b) Let $\{r_n\}$ be an enumeration of the rationals and let $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$. Show g is integrable.
c) Show that g is discontinuous at every point and unbounded on every interval, hence not Riemann integrable on any interval.
d) Show g^2 is finite a.e. but that it is not integrable on any interval.

3. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be complete $\sigma-$finite measure spaces.

a) In $L^p(X, \mathcal{A}, \mu)$ for $p \geq 1$, define $d(f, g) = ||f - g||_p = (\int (f - g)^P d\mu)^{\frac{1}{p}}$. Prove L^p is complete.
b) Express (but do not prove) $||f||_p$ as a supremum of integrals of the form $\int fg d\mu$
c) Let f be a $\mu \times \nu$ measurable function. Prove that $||||f||_{1,\mu}||_{p,\nu} \leq ||||f||_{p_\nu}||_{f,\mu}$.

4. Let $f(x) = x^2 sin(\frac{1}{x})$ and $g(x) = x^2 sin(\frac{1}{x^2})$ for $x \neq 0$ and $f(0) = g(0) = 0$. Show that

a) f and g are differentiable everywhere in $\mathbb{R}$
b) f is of bounded variation on $[-1, 1]$ and g is not of bounded variation on $[-1, 1]$
c) Are f and g absolutely continuous on $[-1, 1]$?

1999 Solutions

1. Prove that if $\{f_n\}$ are non-negative measurable functions that converge a.e. on $\mathbb{R}$ and $\int_{\mathbb{R}} f_n \to \int_{\mathbb{R}} f < \infty$. Then show:

a) $\int_E f = \lim \int_E f_n$ for any measurable sets

Proof. Let $\{f_n\}$ be a sequence of non-negative $\mathcal{F}-$measurable functions converging almost eveywhere to f where $\int f d\mu < \infty$. Let E be any measurable set. By Fatou's lemma we have $\int_E f \leq \underline{\lim} \int_E f_n$. Now we consider $\overline{\lim} \int_E f_n = \overline{\lim}(\int f_n - \int_{E^c} f_n) \leq \overline{\lim}(\int f_n) + \overline{\lim}(-\int_{E^c} f_n) = \int f - \underline{\lim}(\int_{E^c} f_n)$. Again Fatou gives us $\int_{E^c} f \leq \underline{\lim}(\int_{E^c} f_n) \Rightarrow -\int_{E^c} f \geq -\underline{\lim}(\int_{E^c} f_n)$, thus we have $\int f - \underline{\lim}(\int_{E^c} f_n) \leq \int f - \int_{E^c} f = \int_E f$. Thus $\overline{\lim} \int_E f_n \leq \int_E f \Rightarrow \int_E f_n \to \int_E f$. //

b) Show that this needs not be the case if $\int_{\mathbb{R}} f = \infty$

Proof. Consider $X = (0, \infty)$ with the lebesgue measure. Let $f_n = \chi_{(\mathbb{R} \setminus \mathbb{Q}) \cap (0,n)}$ Then we have that $f_n \to f = 1$ a.e. and $\int f_n = n \to \infty = \int 1$. However we have that $\int_{\mathbb{Q}} f_n = 0$ for each n but $\int_{\mathbb{Q}} 1 = \infty$. Therefore $\int_{\mathbb{Q}} f_n$ does not converge to $\int_{\mathbb{Q}} f$. //

2. $f = x^{\frac{-1}{2}} \chi_{(0,1)}$. Then

a) Show f is integrable on $\mathbb{R}$

Proof. Let $f_n = f\chi_{(\frac{1}{n},1)}$. Then $f_n \nearrow f\mu$-a.e, and so by monotone convergence we have $\int f_n \to \int f$. Each f_n is a positive Riemann integrable function and so its lebesgue integral is its Rieamnn integral. Further $R\int_0^1 f = \lim_{n\to\infty} R\int_{\frac{1}{n}}^1 f = \lim_{n\to\infty}(2 - (\frac{1}{n})^{\frac{1}{2}}) = 2 < \infty$. Hence f is integrable. //

b) Let $\{r_n\}$ be an enumeration of the rationals and let $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$. Show g is integrable.

Proof. Since f is non-negative we have that $\int g = \sum_{n=1}^{\infty} 2^{-n} \int f(x - r_n) = \sum_{n=1}^{\infty} 2^{-n+1} = 2 < \infty$. //

c) Show that g is discontinuous at every point and unbounded on every interval, hence not Riemann integrable on any interval.

Proof. Note that for any interval I and $q \in I \cap \mathbb{Q}$ then $\lim_{x\to q^+} f = \infty$ so f is unbounded on I. This shows also that f is discontinuous at every point as supopse g is continuous at some $r \in I$. Then we have that for some δ then $|g(y) - g(r)| < 1$ for $|y - r| < \delta$, at any rational $q \in (r, r + \delta)$ we would have $\overline{\lim_{x\to q}} f < 1$ which is a contradiction. Hence g is not Riemann-integrable on any interval. //

d) Show g^2 is finite a.e. but that it is not integrable on any interval.

Proof. We note that since g is finite a.e. then g^2 is also. However $\int g^2 = \sum_{n=1}^{\infty} 2^{-n} \int f^2(x - r_n)$. Let k be such that $r_k = 0$, then we have that $\int g^2 \geq 2^{-k} \int f^2 = 2^{-k} \int_0^1 \frac{dx}{x} = \infty$. However by arguments above we have that $\int_0^1 \frac{dx}{x} = \lim_{n\to\infty} R\int_{\frac{1}{n}}^1 \frac{dx}{x} = \lim_{n\to\infty} ln(n) = \infty$ So g^2 is not integrable. //

3. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be complete $\sigma-$finite measure spaces.

a) In $L^p(X, \mathcal{A}, \mu)$ for $p \geq 1$, define $d(f, g) = ||f - g||_p = (\int (f - g)^P d\mu)^{\frac{1}{p}}$. Prove L^p is complete.

Proof. It suffices to show that any absolutely summable series in L^p is summable. Give $\sum_{k=1}^{\infty} f_k$ such that $\{f_k\} \subset L^p$, and $\sum_{k=1}^{\infty} ||f_k||_p = M < \infty$, then we will show that $\sum_{k=1}^{\infty} f_k = g \in L^p$ is summable. Consider $h_n = \sum_{k=1}^{n} |f_k|$. Let h be a measurable function such that $h_n \to h \quad \mu-$a.e. We have that $||h_n||_p \leq \sum_{k=1}^{n} ||f_k||_p < M$ by Minkowski's inequality. This gives then that $||h_n||_p^p = \int h_n^p \leq M^p$ and so we have $\int h^p \leq M^p$ which gives that h is finite $\mu-$a.e. Where $h(x) < \infty$ we have that $g(x) = \sum_{k=1}^{\infty} f_k(x)$. Since $\sum_{k=1}^{\infty} f_k$ is absolutely summable we have that $\sum_{k=1}^{\infty} |f_k(x)| < \infty$ and so we have that $\sum_{k=1}^{\infty} f_k(x)$ is a real number. Let $g(x) = 0$ where $h(x) = \infty$. Now let $g_n = \sum_{k=1}^{\infty}$, then we have that $|g_n| \leq |h_n| \leq h$ which gives that $|g| \leq h$ and so $g \in L^p$. Now we have that $g_n \nearrow g \quad \mu-$a.e., and since $|g_n - g|^p \leq 2^{p+1}|g|^p$ we have that $\int |g_n - g|^p \to 0$ by dominated convergence, and so $\sum_{k=1}^{\infty} f_k = g$ is summable, whence L^p is complete. //

b) Express (but do not prove) $||f||_p$ as a supremum of integrals of the form $\int fgd\mu$

Proof. Let q be the conjugate index to p. Then we have that $H(g) = \int gfd\mu$ is a linear functional, and [provided $f \in L^p$] we have that $||H|| = \sup_{||g||_q=1} \int gfd\mu \leq ||f||_p$ is bounded so Riesz Representation gives that $||f||_p = \sup_{||g||_q=1} \int gfd\mu$. //

c) Let f be a $\mu\times\nu$ measurable function. Prove that $||||f||_{1,\mu}||_{p,\nu} \leq ||||f||_{p,\nu}||_{f,\mu}$.

Proof. Consider $h(y) = \int_X |f(x, y)|d\mu$. Let q be the congugate index of p. Then we have that $F : L^q \to \mathbb{R}$ be the functional $F(g) = \int_Y g(y)(\int_X |f(x, y)|d\mu)d\nu$. Then we have that by definition $||F(g)|| = \sup_{||g||_q=1} \int_Y g(y)(\int_X |f(x, y)|d\mu)d\nu$.

And so we have that:

$$\begin{aligned}
||F(g)|| &= \sup_{||g||_q=1} \int_Y g(y)(\int_X |f(x, y)|d\mu)d\nu \\
\text{Tonelli} &= \sup_{||g||_q=1} \int_X (\int_Y g(y)|f(x, y)|d\nu)d\mu \\
\text{Hölder} &\leq \sup_{||g||_q=1} \int_X ||g||_q|| \ |f(x, y)| \ ||_{p,Y} d\mu \\
&\leq \int_X || \ |f(x, y)| \ ||_{p,Y} d\mu
\end{aligned}$$

This being the right hand side. We have also that $\int_X || \, |f(x,y)| \, ||_{p,Y} d\mu < \infty$ otherwise the inequality would be trivially solved. This gives then that F is a bounded linear operator on L^q, and so by Riesz-representation we have that $||F|| = || \int_X |f(x,y)| d\mu ||_{p,Y} \Rightarrow$

$$(\int_Y (\int_X |f(x,y)| d\mu)^p d\nu)^{\frac{1}{p}} \leq \int_X (\int_Y |f(x,y)|^p d\nu)^{\frac{1}{p}} d\mu$$

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4. Let $f(x) = x^2 sin(\frac{1}{x})$ and $g(x) = x^2 sin(\frac{1}{x^2})$ for $x \neq 0$ and $f(0) = g(0) = 0$. Show that

a) f and g are differentiable everywhere in $\mathbb{R}$

Proof. f and g are both differentiable away from 0. So we compute $\lim\limits_{\triangle x \to 0} \frac{g(\triangle x)}{\triangle x} \lim\limits_{\triangle x \to 0} (\triangle x)^2 sin(\frac{1}{(\triangle x)^2)}) = 0$ by the sandwhich theorem, so f is differentiable everywhere. Likewise we compute for f $\lim\limits_{\triangle x \to 0} \frac{f(\triangle x)}{\triangle x} \lim\limits_{\triangle x \to 0} (\triangle x) sin(\frac{1}{(\triangle x))}) = 0$ by the same argument. //

b) f is of bounded variation on $[-1, 1]$ and g is not of bounded variation on $[-1, 1]$

Proof. We have that f is of bounded variation on $[-1, 1]$ since $|f'(x)| = |2x \sin(\frac{1}{x}) + \cos(\frac{1}{x})| \leq 3$ away from zero, and since $f'(0) = 0$ we have that $|f'(x)|$ is bounded on $[-1, 1]$ and hence must be of bounded variation. It will suffice to show that g is not of bounded variation on $[0, 1]$. We first want to consider points $x_k = \sqrt{\frac{2}{(k\pi)}}$. Then we have that $|sin(\frac{1}{x_k^2})| \in \{1, 0\}$. We have then that $T_{-1}^1 g(x) \geq T_0^1 g(x)$. Now since $T_0^1 g(x)$ is larger than the supremum over all partitions $\{0 = x_0, \ldots, x_n = 1\}$ where x_k are taken as above, so we have $T_0^1 g(x) \geq \sum_{k=1}^{\infty} |g(x_k) - g(x_{k-1})| = \sum_{k=1}^{\infty} |g(\sqrt{\frac{2}{k\pi}}) - g(\sqrt{\frac{2}{(k-1)\pi}})| \geq \sum_{k=1}^{\infty} |g(\sqrt{\frac{2}{(2k+1)\pi}}) - g(\sqrt{\frac{2}{2k\pi}})|$, as we are taking few summands. We have then that $\sum_{k=1}^{\infty} |g(\sqrt{\frac{2}{(2k+1)\pi}}) - g(\sqrt{\frac{2}{2k\pi}})| =$
$\sum_{k=1}^{\infty} |\frac{2}{(2k+1)\pi} \sin(\frac{2k+1}{\pi}) - \frac{2}{2k\pi} \sin(\frac{2k}{\pi})| = \sum_{k=1}^{\infty} |\frac{2}{(2k+1)\pi} \sin(\frac{2k+1}{\pi})| =$
$\sum_{k=1}^{\infty} \frac{2}{(2k+1)\pi} = \frac{2}{2k+1} \sum_{k=1}^{\infty} \frac{1}{2k+1} \geq \frac{2}{2k+1} \sum_{k=1}^{\infty} \frac{1}{k}$ which diverges, hence g is not of bounded variation.

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c) Are f and g absolutely continuous on $[-1, 1]$?

Proof. This is clear since f again has a bounded derivative on $[-1, 1]$ and so it is also absolutely continuous. However since we showed that g is not of bounded variation on $[-1, 1]$ and hence cannot be absolutely continuous.

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1998

1. Prove

$$\int_0^1 \frac{x^p}{1-x}\log(\frac{1}{x}) = \sum_{k=1}^{\infty}\frac{1}{(p+k)^2}$$

2. Answer two of the following three questions

a) State the monotone class theorem and use it to show that if $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are finite measure spaces, then the family of subsets E of $X \times Y$ for which the double integrals

$$\int [\int \chi_E d\mu] d\nu = \int [\int \chi_E d\nu] d\mu$$

make sense and are equal is a $\sigma-$algebra that contains the product $\sigma-$algebra $\mathcal{A} \otimes \mathcal{B}$.

b) State the Fubini-Tonelli theorem

c) Use the Fubini-Tonelli theorem to show that if μ is a finite borel measure on $\mathbb{R}$ then

$$\int_0^\infty x^p d\mu = p\int_0^\infty t^{p-1}\mu(x|x > t)dt$$

for all $p > 0$.

3.

a) Prove the Generalized Minkowski inequality, that is prove that if $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are $\sigma-$finite spaces and $f : X \times Y \to \mathbb{R}$ is $\mathcal{A}\otimes\mathcal{B}-$measurable, then $||||f||_{1,\mu}||_{p,\nu} \leq ||||f||_{p,\nu}||_{1,\mu}$ for all $p \geq 1$.

b) Show that if $p > 1$ and $f \in L^p$ then the mean functional of f given by $F(x) = \frac{1}{x}\int_0^x f(y)dy = \int_0^1 f(xt)dt$ is also in L^p and moreover $||F||_p \leq q||f||_p$ where q is the conjugate index of p.

4. Let $T_a^b f$, $P_a^b f$ and $N_a^b f$ be the total, positive and negative variations of f on $[a, b]$.

a) Show that if f is of bounded variation on $[a, b]$, then $\int_a^b |f'(x)|dx \leq T_a^b f$

b) Show that if f is absolutely continuous on $[a, b]$ then $P_a^b f = \int_a^b (f'(x) \vee 0)$, $N_a^b f = -\int_a^b (f'(x) \wedge 0)dx$ and $T_a^b f = \int_a^b |f'(x)|dx$. (Notation $a \wedge b = max(a, b)$ and $a \vee b = min(a, b)$).

1998 Solutions

1. Prove

$$\int_0^1 \frac{x^p}{1-x}\log(\frac{1}{x}) = \sum_{k=1}^{\infty}\frac{1}{(p+k)^2}$$

Proof. We have that $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $|x|<1$, hence almost everywhere on $[0,1]$. This gives that $\int_0^1 \frac{x^p}{1-x}\log(\frac{1}{x}) = \int_0^1 \sum_{k=0}^{\infty} x^{k+p}(-\log(x)) = \int_0^1 \sum_{k=0}^{\infty} -x^{k+p}\log(x)$. Define $f_k = -x^{k+p}\log(x)$ on $(0,1]$ and $f_k(0)=0$, then $\int_0^1 f_k = \int_0^1 -x^{k+p}\log(x)$, and since $f_k \geq 0$ on $[0,1]$ then we have that $\int_0^1 \sum_{k=0}^{\infty} f_k = \sum_{k=0}^{\infty}\int_0^1 f_k$ by monotone convergence. Now take $a_n = \frac{1}{n}$ and $b_n = \frac{n}{n+1}$ then $a_n \to 0$ and $b_n \to 1$ so we have that $\int_0^1 \frac{x^p}{1-x}\log(\frac{1}{x}) = \sum_{k=0}^{\infty}\lim\limits_{n\to\infty}\int_{a_n}^{b_n} f_k = \lim\limits_{n\to\infty}\int_{a_n}^{b_n} -x^{k+p}\log(x)$. On each (a_n,b_n) we have that $-x^{k+p}\log(x)$ is bounded and Riemann integrable and that the lebesgue integral is equal to the Riemann. So we can compute $\int_{a_n}^{b_n} -x^{k+p}\log(x)$ by parts using $u=\log(x)$ and $dv = -x^{k+p}dx$ giving $du = \frac{dx}{x}$ and $v = \frac{-1}{k+p+1}x^{k+p+1}$ and so we have that $\int_{a_n}^{b_n} -x^{k+p}\log(x) = \frac{-x^{k+p+1}\log(x)}{k+p+1}]_{a_n}^{b_n} + \frac{1}{k+p+1}\int_{a_n}^{b_n} x^{k+p} = \frac{-x^{k+p+1}\log(x)}{k+p+1}]_{a_n}^{b_n} + \frac{1}{(1+k+p)^2}x^{k+p+1}]_{a_n}^{b_n} = \frac{-b_n^{k+p+1}\log(b_n)}{k+p+1} + \frac{-a_n^{k+p+1}\log(a_n)}{k+p+1} + (1+k+p)^2 b_n^{k+p+1} - (1+k+p)^2 a_n^{k+p+1}$. Now we take limits as n increases without bound. We have $\lim\limits_{n\to\infty}\frac{-b_n^{k+p+1}\log(b_n)}{k+p+1} = 0$. Further $\lim\limits_{n\to\infty}(1+k+p)^2 b_n^{k+p+1} = \frac{1}{(k+p+1)^2}$, and $\lim\limits_{n\to\infty}(1+k+p)^2 a_n^{k+p+1} = 0$. Now we compute

$\lim\limits_{n\to\infty}\frac{-a_n^{k+p+1}\log(a_n)}{k+p+1}$. If we let $c = k+p+1$ then we are trying to compute the limit $\lim\limits_{n\to\infty}\frac{-a_n^c\log(a_n)}{c} = \frac{-1}{k+p+1}\lim\limits_{n\to\infty} a_n^c\log(a_n) = \frac{-1}{k+p+1}\lim\limits_{n\to\infty}\frac{\log(a_n)}{a_n^{-c}}$. If we attempt to plug in we get the indeterminate form $\frac{\infty}{\infty}$ and so we can use L'Hopitals rule to obtain $\frac{-1}{k+p+1}\lim\limits_{n\to\infty}\frac{a_n^{-1}}{-c\cdot a_n^{-c-1}} = \frac{-1}{k+p+1}\lim\limits_{n\to\infty}\frac{-1}{c\cdot a_n^{-c-1}} = \frac{-1}{k+p+1}\lim\limits_{n\to\infty}\frac{-1}{c}a_n^{c+1} \to 0$. Now putting this all together we obtain $\lim\limits_{n\to\infty}\frac{-b_n^{k+p+1}\log(b_n)}{k+p+1} + \frac{-a_n^{k+p+1}\log(a_n)}{k+p+1} + (1+k+p)^2 b_n^{k+p+1} - (1+k+p)^2 a_n^{k+p+1} = \frac{1}{(k+p+1)^2}$. Hence we have that $\int_0^1 \frac{x^p}{1-x}\log(\frac{1}{x}) = \sum_{k=0}^{\infty}\lim\limits_{n\to\infty}\int_{a_n}^{b_n} f_k = \sum_{k=0}^{\infty}\frac{1}{(k+p+1)^2}$. Reorder by letting $j = k+1$ we have then $\int_0^1 \frac{x^p}{1-x}\log(\frac{1}{x}) = \sum_{j=1}^{\infty}\frac{1}{(j+p)^2}$

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2. Answer two of the following three questions

a) State the monotone class theorem and use it to show that if $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ are finite measure spaces, then the family of subsets E of $X\times Y$ for which the double integrals

$$\int[\int \chi_E d\mu]d\nu = \int[\int \chi_E d\nu]d\mu$$

make sense and are equal is a $\sigma-$algebra that contains the product $\sigma-$algebra $\mathcal{A}\otimes\mathcal{B}$.

Proof. The monotone class theorem essentially states that if a $\sigma-$algebra contains a collection of sets $\mathcal{C}$ then it contains the smallest $\sigma-$algebra which contains the collection $\mathcal{C}$ [11].

Now let $\mathcal{C} = \{E | \int[\int \chi_E d\mu]d\nu = \int[\int \chi_E d\nu]d\mu < \infty\}$. First note that $\int[\int \chi_{X\times Y} d\nu]d\mu = \int \nu(Y)\chi_X d\mu = \mu(X)\nu(Y) = \int \mu(X)\chi_Y d\nu = \int[\int_X \chi_{X\times Y} d\mu]d\nu$. So $X\times Y \in \mathcal{C}$. Now if $E \in \mathcal{C}$ then we have that $X\times Y = \chi_E + \chi_{E^c}$. This gives that $\int[\int \chi_{X\times Y} d\mu]d\nu = \int[\int \chi_E d\mu]d\nu + \int[\int \chi_{E^c} d\mu]d\nu$ and $\int \chi_{X\times Y} d\nu]d\mu = \int[\int \chi_E d\nu]d\mu + \int[\int \chi_{E^c} d\nu]d\mu$ However since $X \times Y$ and E are both in $\mathcal{C}$ we have that $\int[\int \chi_E d\nu]d\mu + \int[\int \chi_{E^c} d\nu]d\mu = \int[\int \chi_E d\mu]d\nu + \int[\int \chi_{E^c} d\mu]d\nu$ which gives that $\int[\int \chi_{E^c} d\mu]d\nu = \int[\int \chi_{E^c} d\nu]d\mu$ and so $E^c \in \mathcal{C}$. Now given a countable disjoint collection $\{E_i\} \in \mathcal{C}$, we have that $\chi_{\bigcup E_i} = \sum \chi_{E_i}$. Then we have that $\int[\int \chi_{\bigcup E_i} d\mu]d\nu = \int[\int \sum \chi_{E_i} d\mu]d\nu = \sum \int[\int \chi_{E_i} d\mu]d\nu = \sum \int[\int \chi_{E_i} d\nu]d\mu = \int[\int \sum \chi_{E_i} d\nu]d\mu = \int[\int \chi_{\bigcup E_i} d\nu]d\mu$. By monotone convergence, thus we have that $\mathcal{C}$ is a $\sigma-$algebra. Since for any measurable rectangle $A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then we have that $\chi_{A\times B}$ is a measurable function. Tonelli's theorem gives that $\int[\int \chi_{A\times B} d\mu]d\nu = \int[\int \chi_{A\times B} d\nu]d\mu$ and so by the monotone class theorem we have that $\mathcal{C}$ contains the algebra $\mathcal{A} \otimes \mathcal{B}$.

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b) State the Fubini-Tonelli theorem

Proof. Fubini and Tonelli are similar theorems. We will state one and note where the other differes. Given $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are $\sigma-$finite spaces, and f be a non-negative $\mathcal{A} \otimes \mathcal{B}-$measurable function. Then we have the following:

(a) $f_x(y) = f(x, y)$ is a measurable function on Y for almost all x
(b) $f^y(x) = f(x, y)$ is a measurable function on X for almost all y
(c) $\int f(x, y)d\nu$ is a measurable function on X
(d) $\int f(x, y)d\mu$ is a measurable function on Y
(e) $\int[\int f d\nu]d\mu = \int[\int f d\mu]d\nu$.

Now Fubini gives essentially the same conclusion with integrable in exchange for measurable. This stronger version comes at the price of requiring our $\mathcal{A} \otimes \mathcal{B}-$measurable function to be integrable and we require our spaces to become complete. //

c) Use the Fubini-Tonelli theorem to show that if μ is a finite borel measure on $\mathbb{R}$ then

$$\int_0^\infty x^p d\mu = p \int_0^\infty t^{p-1}\mu(x|x > t)dt$$

for all $p > 0$.

Proof. Let $f(x, t) = pt^{p-1}\chi_{x>t}$. Then we have that f is measurable in the product space and so we have that $p\int_0^\infty t^{p-1}\mu(x|x > t)dt = \int_0^\infty \int_{\mathbb{R}} pt^{p-1}\chi_{x>t} d\mu dt = \int_0^\infty \int_0^t pt^{p-1} dt d\mu$, By Tonelli. The bounds on the outer integral come from the fact that $f(x, t) = 0$ for $x \le t$. So we have that $p\int_0^\infty t^{p-1}\mu(x|x > t)dt = \int_0^\infty \int_0^t pt^{p-1} dt d\mu = \int_0^\infty x^p d\mu$. //

[11] This theorem was not covered in class and I understand that it is a much more far reaching statement, but to solve the problem this will be sufficient

3.

a) Prove the Generalized Minkowski inequality, that is prove that if $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are $\sigma-$finite spaces and $f : X \times Y \to \mathbb{R}$ is $\mathcal{A} \otimes \mathcal{B}-$measurable, then $||||f||_{1,\mu}||_{p,\nu} \leq ||||f||_{p,\nu}||_{1,\mu}$ for all $p \geq 1$.

Proof. Consider $h(y) = \int_X |f(x,y)|d\mu$. Let q be the congugate index of p. Then we have that $F : L^q \to \mathbb{R}$ be the functional $F(g) = \int_Y g(y)(\int_X |f(x,y)|d\mu)d\nu$. Then we have that by definition $||F(g)|| = \sup_{||g||_q=1} \int_Y g(y)(\int_X |f(x,y)|d\mu)d\nu$.

And so we have that:

$$\begin{aligned}
||F(g)|| &= \sup_{||g||_q=1} \int_Y g(y)(\int_X |f(x,y)|d\mu)d\nu \\
\text{Tonelli} &= \sup_{||g||_q=1} \int_X (\int_Y g(y)|f(x,y)|d\nu)d\mu \\
\text{Hölder} &\leq \sup_{||g||_q=1} \int_X ||g||_q|| \ |f(x,y)| \ ||_{p,Y} d\mu \\
&\leq \int_X || \ |f(x,y)| \ ||_{p,Y} d\mu
\end{aligned}$$

This being the right hand side. We have also that $\int_X || \ |f(x,y)| \ ||_{p,Y} d\mu < \infty$ otherwise the inequality would be trivially solved. This gives then that F is a bounded linear operator on L^q, and so by Riesz-representation we have that $||F|| = || \int_X |f(x,y)|d\mu||_{p,Y} \Rightarrow$

$$(\int_Y (\int_X |f(x,y)|d\mu)^p d\nu)^{\frac{1}{p}} \leq \int_X (\int_Y |f(x,y)|^p d\nu)^{\frac{1}{p}} d\mu$$

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b) Show that if $p > 1$ and $f \in L^p$ then the mean functional of f given by $F(x) = \frac{1}{x} \int_0^x f(y)dy = \int_0^1 f(xt)dt$ is also in L^p and moreover $||F||_p \leq q||f||_p$ where q is the conjugate index of p.

Proof. Consider $||F||_p = (\int_X (\int_0^1 f(xt)dt)^p d\mu)^{\frac{1}{p}} \leq \int_0^1 (\int_X f(xt)^p d\mu)^{\frac{1}{p}} dt$ by general minkowski. By substitution $u = xt$ we have $du = td\mu$. Then we have $\int_0^1 (\int_X f(xt)^p d\mu)^{\frac{1}{p}} dt = \int_0^1 (\int_X f(u)^p td\mu)^{\frac{1}{p}} dt = \int_0^1 (\int_X f(u)^p td\mu)^{\frac{1}{p}} dt = \int_0^1 ||f||_p t^{\frac{1}{p}} dt = \frac{1}{\frac{1}{p}+1}||f||_p = \frac{p}{p+1}||f||_p \leq \frac{p}{p-1}||f||_p = q||f||_p$

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4. Let $T_a^b f$, $P_a^b f$ and $N_a^b f$ be the total, positive and negative variations of f on $[a, b]$.

a) Show that if f is of bounded variation on $[a, b]$, then $\int_a^b |f'(x)|dx \leq T_a^b f$

Proof. We have that since f is of bounded variation then we can write $f = g - h$ where $g(x) = P_a^x$ and $h(x) = N_a^x$. Then we have also that f is differentiable almost everywhere, then we have that $\int_a^b |f'(x)| = \int_a^b |g'(x) - h'(x)| \leq \int_a^b |g'(x)| + \int_a^b |h'(x)|$ Now we have that $\int_a^b |g'(x)| \leq g(b) - g(a)$ and

$\int_a^b |h'(x)| \leq h(b)-h(a)$ and so we have $\int_a^b |f'(x)| \leq g(b)-g(a)+h(b)-h(a) = P_a^b f + N_a^b f = T_a^b f$. //

b) Show that if f is absolutely continuous on $[a, b]$ then $P_a^b f = \int_a^b (f'(x) \vee 0)$, $N_a^b f = -\int_a^b (f'(x) \wedge 0)dx$ and $T_a^b f = \int_a^b |f'(x)|dx$. (Notation $a \wedge b = max(a, b)$ and $a \vee b = min(a, b)$).

Proof. We already have that $\int_a^b |f'(x)| \leq T_a^b$. It suffices to show the reverse inequality. If f is absolutely continuous then we have that then we have that $T_a^b = \sup_{P \text{ partition}} \sum_{k=1}^n |f(x_i) - f(x_{i-1})| = \sup_{P \text{ partition}} \sum_{k=1}^n |\int_{x_i}^{x_{i-1}} f'(x)| = \sup_{P \text{ partition}} |\int_a^b f'(x)| = |\int_a^b f'(x)| \leq \int_a^b |f'(x)|$. It suffices to show that $P_a^b = \int_a^b (f'(x) \vee 0)$, since $T_a^b f = P_a^b f + N_a^b f$. Now from part (a) we have that $\int_a^b (f'(x) \vee 0) \leq P_a^b f$. And as before we compute $P_a^b = \sup_{P \text{ partition}} \sum_{k=1}^n [f(x_i) - f(x_{i-1})]^+ = \sup_{P \text{ partition}} \sum_{k=1}^n [\int_{x_i}^{x_{i-1}} f'(x)]^+ = \sup_{P \text{ partition}} [\int_a^b f'(x)]^+$ Since on any interval $[x_{i-1}, x_i]$ such that $\int_{x_i}^{x_{i-1}} f'(x) \leq 0$ then we have that $f' \leq 0$ almost everywhere and so $[\int_a^b f'(x)]^+ = \sum_{k=1}^\infty [\int_{x_i}^{x_{i-1}} f'(x)]^+$. Now we have that $\sup_{P \text{ partition}} [\int_a^b f'(x)]^+ = [\int_a^b f'(x)]^+ \leq \int_a^b [f'(x)]^+$ by a similar argument and so $P_a^b = \int_a^b [f'(x)]^+$. //

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Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA
E-mail address: lmiller@math.uconn.edu