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Statistics 100A **Instructor: Nicolas Christou**

Joint probability distributions

So far we have considered only distributions with one random variable. There are many problems that involve two or more random variables. We need to examine how these random variables are distributed together ("jointly"). There are discrete and continuous joint distributions.

Discrete:
Here is an Example:

Let X be the number of puppies born, and Y be the number of puppies survived for a certain breed of dog. Suppose the following table describes the distribution of X and Y:

		X			
		3	4	5	
	0	0.31	0.21	0.21	0.73
Y	1	0.03	0.04	0.05	0.12
	2	0.02	0.03	0.04	0.09
	3	0.01	0.02	0.03	0.06
		0.37	0.30	0.33	1.0

Notation:

- Joint (or bivariate) probability distribution of X and Y:
  $f_{XY}(x, y) = P(X = x, Y = y)$
  Example:

- Always

$$\sum_x \sum_y f_{XY}(x, y) = 1$$

- Marginal distribution of X:

$$f_X(x) = \sum_y f_{XY}(x, y)$$

Example:

- Marginal distribution of Y:

$$f_Y(y) = \sum_x f_{XY}(x, y)$$

Example:

- Conditional probability function:

$$f_{Y|X}(y|x) = P(Y = y|X = x) = \frac{P(Y = y, X = x)}{P(X = x)}$$

Or

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

Example:

- Joint cumulative distribution function:

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

Example:

- Conditional expectation:

$$E[Y|X = x] = \sum_y y f_{Y|X}(y|x)$$

Example:

- Conditional variance:

$$var[Y|X = x] = E[Y^2|X = x] - (E[Y|X = x])^2$$

or

$$var[Y|X = x] = \sum_y y^2 f_{Y|X}(y|x) - \left(\sum_y y f_{Y|X}(y|x)\right)^2$$

Example:

- Interesting and important property of conditional expectation:

$$E(Y) = E\left[E[Y|X]\right]$$

- Independent discrete random variables:
  X and Y are independent if

$$P(X = x, Y = y) = P(X = x)P(Y = y), \text{ for all pairs of } x, y.$$

- Expected value of a function of X and Y:

$$E\left[g_{X,Y}(x,y)\right] = \sum_x \sum_y g_{X,Y}(x,y) f_{XY}(x,y)$$

Joint distributions - Examples

(From Sheldon Ross (2006), *A First Course in Probability*, 7th Edition, Prentice Hall).

Example 1:
A die is rolled and the number observed X is recorded. Then a coin is tossed number of times equal to the value of X. For example if $X = 2$ then the coin is tossed twice, etc. Let Y be the number of heads observed. Note: Assume that the die and the coin are fair.

a. Construct the joint probability distribution of X and Y.
b. Find the conditional expected value of Y given $X = 5$.
c. Find the conditional variance of Y given $X = 5$.
d. Find the expected value of Y.
e. Find the standard deviation of Y.
f. Graph the probability distribution of Y.

Example 2:
A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours of travel. If we assume that the miner is at all times equally likely to choose any of the doors, what is the expected length of time until he reaches safety?

Example 3:
The game of craps is begun by rolling an ordinary pair of dice. If the sum of the dice is 2, 3, or 12, the player loses. If it is 7 or 11, the player wins. If it is any other number ($i = 4, 5, 6, 8, 9, 10$), the player continues to roll until the sum is either 7 or i. If it is 7, the player loses; if it is i, the player wins. Let R denote the number of rolls of the dice until a game of craps stops. Find $E(R)$ by conditioning on S, the initial sum. In other words use: $E(R) = E[E(R|S)]$.

- **Continuous:**
- $f_{XY}(x,y)$ is said to be the joint probability density function of X and Y if

$$\int_y \int_x f_{XY}(x,y)dxdy = 1$$

Example: $f_{XY}(x,y) = 1, 0 \leq x \leq 1, 0 \leq y \leq 1.$

a. Sketch this density.

b. Find $P(0 \leq x \leq 0.1, 0 \leq y \leq 0.8)$

- Marginal probability density function of X:

$$f_X(x) = \int_y f_{XY}(x,y)dy$$

- Marginal probability density function of Y:

$$f_Y(y) = \int_x f_{XY}(x,y)dx$$

Example: Let $f_{XY}(x,y) = 2x, 0 \leq x \leq 1, 0 \leq y \leq 1.$
Find the marginal pdf of X.

Find the marginal pdf of Y.

- Conditional probability density function of X given Y:

$$f_{X/Y}(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

- Conditional probability density function of Y given X:

$$f_{Y/X}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

Example: Let $f_{XY}(x, y) = 21x^2y^3, 0 < x < y < 1$.

a. Find marginal pdf of Y.

b. Find conditional pdf of X given Y.

c. Find conditional expectation of X given Y.

Example: Let $f_{XY}(x,y) = 2, 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x + y \leq 1$

a. Find $P(X \geq \frac{1}{2} | Y \leq \frac{1}{4})$.

b. Find $P(X \geq \frac{1}{2} | Y = \frac{1}{4})$.

The Bivariate Normal Distribution

The bivariate normal distribution for two normally distributed random variables $X \sim N(\mu_X, \sigma_X)$ and $Y \sim N(\mu_Y, \sigma_Y)$ is defined as follows:

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} exp\left[-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{x-\mu_y}{\sigma_y}\right)\right]\right]$$

with

$$\int_y \int_x f_{XY}(x,y)dxdy = 1,$$

where ρ is the correlation coefficient between X, Y.