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Statistics 100A **Instructor: Nicolas Christou**

Moment generating functions

Definition:

$$M_X(t) = Ee^{tX}$$

Therefore,
If X is discrete

$$M_X(t) = \sum_x e^{tX} P(x)$$

If X is continuous

$$M_X(t) = \int_x e^{tX} f(x)dx$$

Aside:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Similarly,

$$e^{tx} = 1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots$$

Let X be a discrete random variable.

$$M_X(t) = \sum_x e^{tX} P(x) = \sum_x \left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots\right] P(x)$$

or

$$M_X(t) = \sum_x P(x) + \frac{t}{1!}\sum_x xP(x) + \frac{t^2}{2!}\sum_x x^2 P(x) + \frac{t^3}{3!}\sum_x x^3 P(x) + \cdots$$

To find the k_{th} moment simply evaluate the k_{th} derivative of the $M_X(t)$ at $t = 0$.

$$EX^k = [M_X(t)]_{t=0}^{k_{th}}\text{derivative}$$

For example:
First moment:

$$M_X(t)' = \sum_x xP(x) + \frac{2t}{2!}\sum_x x^2 P(x) + \cdots$$

We see that $M_X(0)' = \sum_x xP(x) = E(X)$.

Similarly,
Second moment

$$M_X(t)'' = \sum_x x^2 P(x) + \frac{3t^2}{3!} \sum_x x^3 P(x) + \cdots$$

We see that $M_X(0)'' = \sum_x x^2 P(x) = E(X^2)$.

Examples:
Find the moment generating function of $X \sim b(n, p)$.
Find the moment generating function of $X \sim Poisson(\lambda)$.
Find the moment generating function of $X \sim exp(\lambda)$.
Find the moment generating function of $Z \sim N(0, 1)$.

Theorem:
Let X, Y be independent random variables with moment generating functions $M_X(t), M_Y(t)$ respectively. Then, the moment generating function of the sum of these two random variables is equal to the product of the individual moment generating functions:

$$M_{X+Y}(t) = M_X(t)M_Y(t))$$

Proof:

Use this theorem to find the distribution of $X + Y$, where $X \sim b(n_1, p), Y \sim b(n_2, p)$.
Use this theorem to find the distribution of $X+Y$, where $X \sim Poisson(\lambda_1), Y \sim Poisson(\lambda_2)$.
Use this theorem to find the distribution of $X + Y$, where $X \sim N(\mu_1, \sigma_1), Y \sim N(\mu_2, \sigma_2)$.

Properties of moment generating functions:
Let X be a random variable with moment generating function $M_X(t) = Ee^{tX}$, and a, b are constants

1. $M_{X+a}(t) = e^{at} M_X(t)$
2. $M_{bX}(t) = M_X(bt)$
3. $M_{\frac{X+a}{b}} = e^{\frac{a}{b}t} M_X(\frac{t}{b})$

Proof:

Use these properties and the moment generating function of $Z \sim N(0, 1)$ to find the moment generating function of $X \sim N(\mu, \sigma)$