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Statistics 100A **Instructor: Nicolas Christou**

Poisson, Gamma, and Exponential distributions

- **A. Relation of Poisson and exponential distribution:**
  Suppose that events occur in time according to a Poisson process with parameter λ. So $X \sim Poisson(\lambda)$. Let T denote the length of time until the first arrival. Then T is a continuous random variable. To find the probability density function (pdf) of T we begin with the cumulative distribution function (cdf) of T as follows:

  $$F(t) = P(T \leq t) = 1 - P(T > t) = 1 - P(X = 0)$$

  In words: The probability that we observe the first arrival after time t is the same as the probability that we observe no arrivals from now until time t. But X is Poisson with parameter λ which has parameter λt over the time interval $(0, t)$. We compute the above using:

  $$F(t) = 1 - \frac{(\lambda t)^0 e^{-\lambda t}}{0!} \Rightarrow F(t) = 1 - e^{-\lambda t}.$$

  To find the pdf pf T we take the derivative of the cdf w.r.t. t to get:

  $$f(t) = F(t)' = \lambda e^{-\lambda t}.$$

  We observe that if $X \sim Poisson(\lambda)$ the time until the first arrival is exponential with parameter λ.

  Example:
  Suppose that an average of 20 customers per hour arrive at a shop according to a Poisson process ($\lambda = \frac{1}{3}$ per minute). What is the probability that the shopkeeper will wait more than 5 minutes before the first customer arrives?

- **B. Relation of Poisson and gamma distribution:**
Suppose that events occur in time according to a Poisson process with parameter λ. So $X \sim Poisson(\lambda)$. Let T denote the length of time until k arrivals. Then T is a continuous random variable. To find the probability density function (pdf) of T we begin with the cumulative distribution function (cdf) of T as follows:

$$F(t) = P(T \leq t) = 1 - P(T > t) = 1 - P(X < k) = 1 - P(X \leq k-1)$$

In words: The probability that we observe the k_{th} arrival after time t is the same as the probability that we observe less that k arrivals from now until time t. But X is Poisson with parameter λ which has parameter λt over the time interval $(0, t)$. We compute the above using:

$$F(t) = 1 - P(X \leq k-1) \quad = \quad 1 - \sum_{x=0}^{k-1} \frac{(\lambda t)^x e^{-\lambda t}}{x!} = 1 - e^{-\lambda t} \sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!}$$

To find the pdf pf T we take the derivative of the cdf w.r.t. t to get:

$$\begin{aligned}
f(t) &= F(t)' \\
&= e^{-\lambda t}\lambda \sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!} - e^{-\lambda t} \sum_{x=0}^{k-1} \frac{x(\lambda t)^{x-1}\lambda}{x!} \\
&= e^{-\lambda t}\lambda \left[\sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!} - \sum_{x=1}^{k-1} \frac{x(\lambda t)^{x-1}}{x!}\right] \\
&= e^{-\lambda t}\lambda \left[\sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!} - \sum_{x=1}^{k-1} \frac{x(\lambda t)^{x-1}}{x(x-1)!}\right] \\
&= e^{-\lambda t}\lambda \left[\sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!} - \sum_{x=1}^{k-1} \frac{(\lambda t)^{x-1}}{(x-1)!}\right], \text{ for the second term let } y = x - 1 \\
&= e^{-\lambda t}\lambda \left[\sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!} - \sum_{y=0}^{k-2} \frac{(\lambda t)^y}{y!}\right].
\end{aligned}$$

The square bracket is reduced to $\frac{(\lambda t)^{k-1}}{(k-1)!}$ because,

$$\left[\sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!} - \sum_{y=0}^{k-2} \frac{(\lambda t)^y}{y!}\right] =$$

$$1 + \frac{(\lambda t)}{1!} + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \cdots + \frac{(\lambda t)^{k-2}}{(k-2)!} + \frac{(\lambda t)^{k-1}}{(k-1)!} -$$

$$1 - \frac{(\lambda t)}{1!} - \frac{(\lambda t)^2}{2!} - \frac{(\lambda t)^3}{3!} - \cdots - \frac{(\lambda t)^{k-2}}{(k-2)!}.$$

So far we have

$$f(t) = e^{-\lambda t}\lambda\frac{(\lambda t)^{k-1}}{(k-1)!} = \frac{t^{k-1}\lambda^k e^{-\lambda t}}{(k-1)!}$$

But, since k is an integer (number of k arrivals), $\Gamma(k) = (k-1)!$. The expression above can be written as:

$$f(t) = \frac{t^{k-1}\lambda^k e^{-\lambda t}}{\Gamma(k)}$$

Compare $f(t)$ with the gamma pdf:

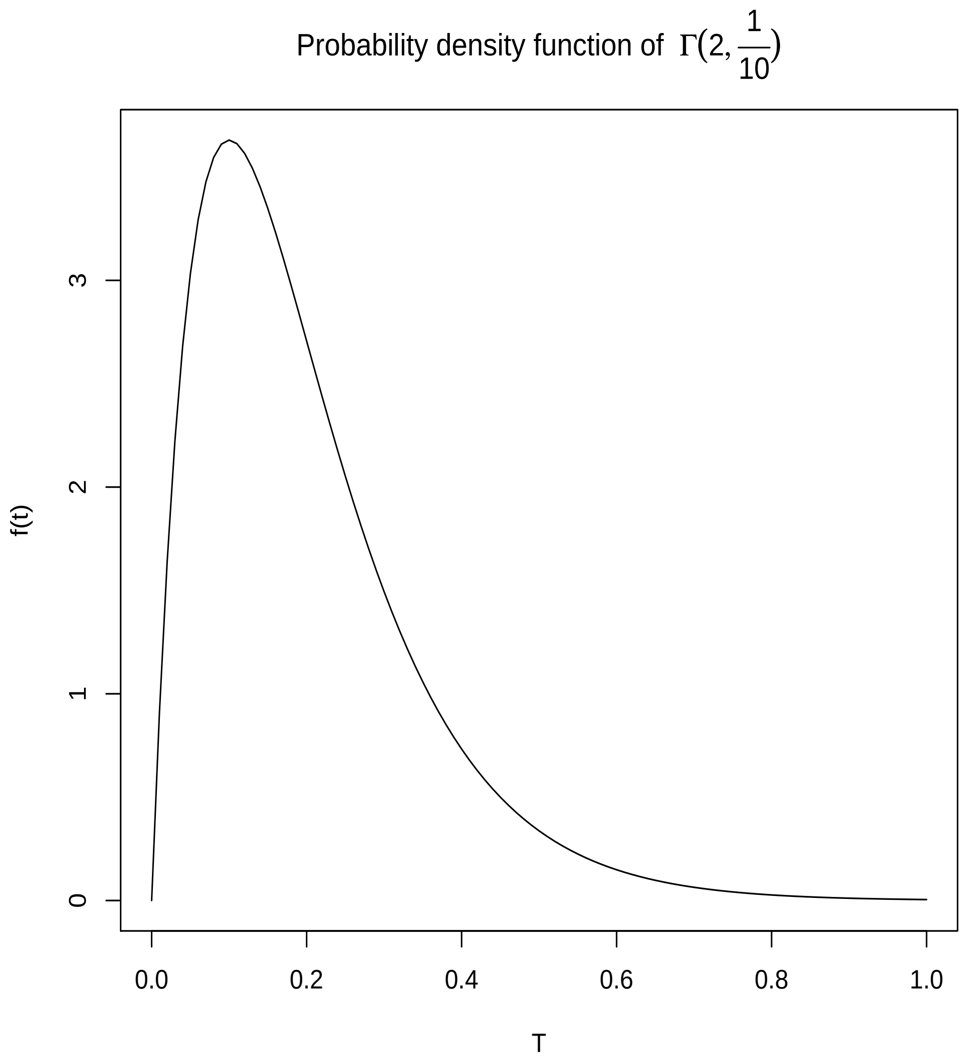
$$f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha\Gamma(\alpha)}, \quad \alpha, \beta > 0, x \geq 0.$$

We observe that $f(t)$ is the density of a gamma distribution with parameters $\alpha = k$ and $\beta = \frac{1}{\lambda}$.
Conclusion: If $X \sim Poisson(\lambda)$ the time until k arrivals is $\Gamma(k, \frac{1}{\lambda})$.

Example:
Suppose customers arrive at a store as a Poisson process with $\lambda = 10$ customers per hour.

a. What is the distribution of the time until the second customer arrives (see graph on next page)?
b. Find the probability that one has to wait at least half an hour until the second customer arrives.

Part (a):



The graph above was constructed in `R`:

```
> t <- seq(0,1,0.01)
> ft <- 100*t*exp(-10*t)
> plot(t,ft,type="l", xlab="T", ylab="f(t)")
> title(main=expression(paste("Probability density function of  ",
  Gamma(2,frac(1,10)))))
```

Part (b):
We want to compute

$$P(T \geq \frac{1}{2}) = \int_{\frac{1}{2}}^{\infty} 100te^{-10t}dt$$

Using integration by parts:
$u = t \Rightarrow du = dt$ and $dv = e^{-10t}dt \Rightarrow v = -\frac{1}{10}e^{-10t}$.

$$\begin{aligned} P(T \geq \frac{1}{2}) &= \int_{\frac{1}{2}}^{\infty} 100te^{-10t}dt \\ &= 100\left[-\frac{1}{10}e^{-10t}t - \int_{\frac{1}{2}}^{\infty} -\frac{1}{10}e^{-10t}dt\right] \\ &= 100\left[-\frac{1}{10}e^{-10t}t - \frac{1}{100}e^{-10t}\right]_{\frac{1}{2}}^{\infty} \\ &= 100\left[\frac{1}{20}e^{-5} + \frac{1}{100}e^{-5}\right] = 6e^{-5} = 0.04042768. \end{aligned}$$

Using R this probability can be computed as follows:

```
pgamma(0.5,2,10, lower.tail=FALSE)
[1] 0.04042768
```