APPENDIX E

Riemann-Stieltjes Integrals

Recall : Consider the Riemann integral

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i) \qquad t_i \in [x_i, x_{i+1}].$$

Consider the expectation introduced in Chapter 1,

$$\mathbb{E}[X] = \int_\Omega X\,d\mathbb{P} = \int_{-\infty}^{\infty} x\,dF(x) = \int_{-\infty}^{\infty} xp(x)\,dx, \tag{E.1}$$

where p is the probability density function of X, and F is the cumulative distribution function of X. The second integral in (E.1) is the Lebesgue integral, the fourth in (E.1) is the Riemann integral. What is the third integral in (E.1)?

E.1. Definition

Basic Assumptions: The functions f, g, α, β are bounded on $[a, b]$.

Definition E.1. Let $P = \{x_1, x_2, \cdots, x_n\}$ be a partition of $[a, b]$ and $t_k \in [x_{k-1}, x_k]$ for $k = 1, 2, \cdots, n$.

(1) A sum of the form

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)(\alpha(x_k) - \alpha(x_{k-1}))$$

is called a Riemann-Stieltjes sum of f with respect to α.

(2) A function f is Riemann-Stieltjes Integrable with respect to α on $[a,b]$, and we write "$f \in R(\alpha)$ on $[a,b]$", if there exists $A \in \mathbb{R}$ such that

$$S(P,f,\alpha) \longrightarrow A \qquad \text{as} \quad \max_k |x_k - x_{k-1}| \longrightarrow 0.$$

也就是說分割地愈細, $S(P,f,\alpha)$ 會愈接近 A.

Notation E.2. If the number A exists in Definition E.1(2), it is uniquely determined and is denoted by

$$\int_a^b f\,d\alpha \qquad \text{or} \qquad \int_a^b f(x)\,d\alpha(x).$$

We also say that the Riemann-Stieltjes Integral $\int_a^b f\,d\alpha$ exists.

Example E.3. Let $f(x) = x$, and $\alpha(x) = x + [x]$. Find $\int_0^{10} f(x)\,d\alpha(x)$.

Solution. Consider the partition $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{10n}{n}\right\}$. Then

$$\begin{aligned} S(P,f,\alpha) &= \sum_{k=1}^{10n} f(t_k)\left(\alpha\left(\frac{k}{n}\right) - \alpha\left(\frac{k-1}{n}\right)\right) \\ &= \sum_{k=1}^{10n} t_k\left(\left(\frac{k}{n} + \left[\frac{k}{n}\right]\right) - \left(\frac{k-1}{n} + \left[\frac{k-1}{n}\right]\right)\right) \\ &= \sum_{k=1}^{10n} t_k\left(\frac{1}{n} + \left(\left[\frac{k}{n}\right] - \left[\frac{k-1}{n}\right]\right)\right) \\ &= \sum_{k=1}^{10n} \frac{t_k}{n} + \sum_{k=1}^{10n} t_k\left(\left[\frac{k}{n}\right] - \left[\frac{k-1}{n}\right]\right). \end{aligned}$$

Since

$$\sum_{k=1}^{10n} \frac{t_k}{n} \longrightarrow \int_0^{10} x\,dx = \left.\frac{x^2}{2}\right|_{x=0}^{10} = 50,$$

and

$$\sum_{k=1}^{10n} t_k\left(\left[\frac{k}{n}\right] - \left[\frac{k-1}{n}\right]\right) = \sum_{i=0}^{9} t_{(i+1)n}((i+1) - i) \longrightarrow 55,$$

as $n \to \infty$, we have

$$\int_0^{10} f(x)\,d\alpha(x) = 50 + 55 = 105.$$

E.2. Properties

Theorem E.4. *Let c_1, c_2 be two constants in $\mathbb{R}$.*

(1) *If $f, g \in R(\alpha)$ on $[a,b]$, then $c_1 f + c_2 g \in R(\alpha)$ on $[a,b]$, and*

$$\int_a^b (c_1 f + c_2 g)\,d\alpha = c_1 \int_a^b f\,d\alpha + c_2 \int_a^b g\,d\alpha.$$

(2) *If $f \in R(\alpha)$ and $f \in R(\beta)$ on $[a,b]$, then $f \in R(c_1\alpha + c_2\beta)$ on $[a,b]$, and*

$$\int_a^b f\,d(c_1\alpha + c_2\beta) = c_1 \int_a^b f\,d\alpha + c_2 \int_a^b f\,d\beta.$$

(3) *If $c \in [a,b]$, then*

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha.$$

Definition E.5. If $a < b$, we define

$$\int_b^a f\,d\alpha = -\int_a^b f\,d\alpha.$$

Theorem E.6. *If $f \in R(\alpha)$ and α has a continuous derivative on $[a,b]$, then the Riemann integral $\int_a^b f(x)\alpha'(x)\,dx$ exists and*

$$\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\alpha'(x)\,dx.$$

E.3. Technique of integrations

E.3.1. Integration by parts.

Theorem E.7 (Integration by parts). *If $f \in R(\alpha)$ on $[a,b]$, then $\alpha \in R(f)$ on $[a,b]$, and*

$$\int_a^b f(x)\,d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)\,df(x).$$

Example E.8. As in Example E.3, $f(x) = x$, and $\alpha(x) = x + [x]$. Then

$$\begin{aligned}\int_0^{10} f(x)\,d\alpha(x) &= f(10)\alpha(10) - f(0)\alpha(0) - \int_0^{10} \alpha(x)\,df(x)\\ &= 10 \times 20 - 0 \times 0 - \int_0^{10} (x + [x])\,dx\\ &= 200 - 50 - \int_0^{10} [x]\,dx = 150 - 45 = 105\end{aligned}$$

E.3.2. Change of variables.

Theorem E.9 (Change of variables). *Suppose that $f \in R(\alpha)$ on $[a,b]$ and g is a strictly increasing continuous function on $[c,d]$ with $a = g(c)$, $b = g(d)$. Let $h = f \circ g$, $\beta = \alpha \circ g$. Then $h \in R(\beta)$ on $[c,d]$ and*

$$\int_a^b f(x)\,d\alpha(x) = \int_c^d f(g(t))\,d\alpha(g(t)) = \int_c^d h(t)\,d\beta(t).$$

Example E.10. Let $y = \sqrt{x}$, we have

$$\begin{aligned}\int_0^4 ([\sqrt{x}] + x^2)\,d\sqrt{x} &= \int_0^2 ([y] + y^4)\,dy = \int_0^2 [y]\,dy + \int_0^2 y^4\,dy\\ &= 1 + \frac{1}{5}y^5\Big|_{y=0}^{2} = \frac{37}{5}\end{aligned}$$

E.3.3. Step functions as α. By Remark C.6 and Theorem E.4(2), we have

$$\int_a^b f(x)\,dF(x) = \int_a^b f(x)\,dF_{ac}(x) + \int_a^b f(x)\,dF_{sc}(x) + \int_a^b f(x)\,dF_d(x) \tag{E.2}$$

其中 $\int_a^b f(x)\,dF_{ac}(x)$ 可利用 Theorem E.6 改成 Riemann integral. 在這一小節我們有興趣的是討論 $\int_a^b f(x)\,dF_d(x)$ 這個積分.

Remark E.11. If $\alpha \equiv$ constant on $[a,b]$, then $S(P,f,\alpha)=0$ for all partition P, and

$$\int_a^b f(x)\,d\alpha(x) = 0.$$

我們現在有興趣的是 α 為 step functions 時的積分.

Theorem E.12. *Given $c \in (a,b)$. Define*

$$\alpha(x) = pI_{[a,c)} + rI_{\{c\}} + qI_{(c,b]}$$

(as given in Figure E.1). Suppose at least one of the functions f or α is continuous from the left at c, and at least one is continuous from the right at c. Then $f \in R(\alpha)$ and

$$\int_a^b f(x)\,d\alpha(x) = f(c)(\alpha(c+)-\alpha(c-)) = f(c)(q-p).$$[1]

Remark E.13. The integral $\int_a^b f\,d\alpha$ does not exist if both of f and α are discontinuous from the left or from the right at c.

Remark E.14. (1) If $\alpha(x) = pI_{\{a\}} + qI_{(a,b]}$, then

$$\int_a^b f(x)\,d\alpha(x) = f(a)(\alpha(a+)-\alpha(a))$$

[1]Note that this value is independent of the value of $\alpha(c)$.

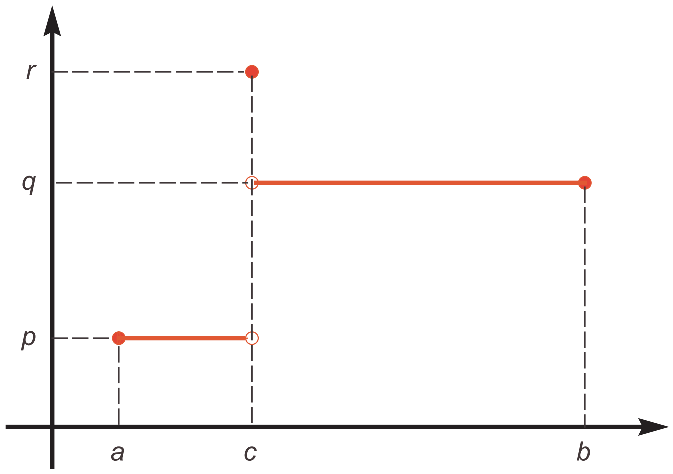

FIGURE E.1. The simple function α.

(2) If $\alpha(x) = pI_{[a,b)} + qI_{\{b\}}$, then

$$\int_a^b f(x)\,d\alpha(x) = f(b)(\alpha(b) - \alpha(b-))$$

Example E.15. (1) Consider

$$f(x) = 1 \quad \text{for } x \in [-1,1], \qquad \text{and} \qquad \alpha(x) = -I_{\{0\}},$$

then

$$\int_{-1}^1 f(x)\,d\alpha(x) = f(0)(\alpha(0+) - \alpha(0-)) = 0$$

(2) Consider

$$f(x) = 2I_{\{0\}} + I_{[-1,0)\cup(0,1]} \qquad \text{and} \qquad \alpha(x) = -I_{[0,1]}.$$

Then both of α and f are discontinuous from the left at $x = 0$. This implies that the Riemann-Stieltjes integral $\int_{-1}^1 f\,d\alpha$ does not exist.

Theorem E.16 (Reduction of a Riemann-Stieltjes Integral to a finite sum). *Let α be a step function on $[a, b]$ with jump*

$$c_k = \alpha(x_k+) - \alpha(x_k-) \quad \text{at } x = x_k.$$

Let f be defined on $[a, b]$ in such a way that not both of f and α are discontinuous from the left or from the right at x_k. Then $\int_a^b f(x)\, d\alpha(x)$ exists and

$$\int_a^b f(x)\, d\alpha(x) = \sum_{k=1}^{n} f(x_k) c_k.$$

Example E.17. (1) Let

$$f(x) = \begin{cases} 3 & \text{if } x \leq 0 \\ 3 - 4x & \text{if } 0 < x < 1 \\ -1 & \text{if } x \geq 1 \end{cases}$$

and

$$\alpha(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x \geq 1 \end{cases}$$

Since f is continuous, $\int_{-3}^{3} f(x)\, d\alpha(x)$ exists and

$$\begin{aligned} \int_{-3}^{3} f(x)\, d\alpha(x) &= f(0)(\alpha(0+) - \alpha(0-)) + f(1)(\alpha(1+) - \alpha(1-)) \\ &= 3(2 - 0) + (-1)(0 - 2) = 8. \end{aligned}$$

(2) Let $\alpha(x) = 2I_{[0,2)} + 5I_{[2,3)} + 6I_{[3,\infty)}$

$$\begin{aligned} \int_{-5}^{10} e^{-3x}\, d\alpha(x) &= e^{-3\cdot 0}(2 - 0) + e^{-3\cdot 2}(5 - 2) + e^{-3\cdot 3}(6 - 5) \\ &= 2 + 3e^{-6} + e^{-9}. \end{aligned}$$

在這節的最後, 我們看看一個 $\int_a^b f(x)\,dF_{sc}(x)$ 的例子.

Example E.18. Suppose F is the Cantor function (see Figure C.1). By integration by parts, we have

$$\int_0^1 x\,dF(x) = xF(x)|_{x=0}^1 - \int_0^1 F(x)\,dx = 1 - \int_0^1 F(x)\,dx.$$

Since $\int_0^1 F(x)\,dx$ is the area of the Cantor function on $[0,1]$, we get

$$\int_0^1 F(x)\,dx = \frac{1}{2}.$$

Hence,

$$\int_0^1 x\,dF(x) = \frac{1}{2}.$$

E.3.4. Comparison theorem.

Theorem E.19. *Assume that α is an increasing function on $[a,b]$. If $f, g \in R(\alpha)$ on $[a,b]$, and if $f(x) \le g(x)$ for $x \in [a,b]$, then*

$$\int_a^b f(x)\,d\alpha(x) \le \int_a^b g(x)\,d\alpha(x).$$

Corollary E.20. *If $g(x) \ge 0$ and α is an increasing function on $[a,b]$, then*

$$\int_a^b f(x)\,d\alpha(x) \ge 0.$$

Theorem E.21. *Assume that α is an increasing function on $[a,b]$. If $f \in R(\alpha)$ on $[a,b]$, then*

(1) $|f| \in R(\alpha)$ *on* $[a,b]$, *and*

$$\left|\int_a^b f(x)\,d\alpha(x)\right| \le \int_a^b |f(x)|\,d\alpha(x).$$

(2) $f^2 \in R(\alpha)$ *on* $[a,b]$.

Theorem E.22. *Assume that* α *be an increasing function on* $[a, b]$*. If* $f, g \in R(\alpha)$ *on* $[a, b]$*, then* $f \cdot g \in R(\alpha)$*.*

E.4. Bounded variation and Riemann-Stieltjes integral

Definition E.23. A function $\alpha : [a, b] \longrightarrow \mathbb{R}$ is said to be of bounded variation if there exists a constant M such that

$$\sum_{k=1}^{n} |\alpha(x_k) - \alpha(x_{k-1})| \leq M$$

for every partition $\{x_0, x_1, \cdots, x_n\}$ of $[a, b]$.

Bounded variation 說穿了就是講函數上下震盪總和為 bounded. 但哪些函數會是 of bounded variation?

Theorem E.24. *Let* α *be defined on* $[a, b]$*, then* α *is of bounded variation on* $[a, b]$*, if and only if there exist two increasing functions* α_1 *and* α_2*, such that* $\alpha = \alpha_1 - \alpha_2$

Theorem E.25. *If* f *is continuous on* $[a, b]$*, and if* α *is of bounded variation on* $[a, b]$*, then* $f \in R(\alpha)$*. Moreover, the function*

$$F(t) = \int_0^t f(x)\, d\alpha(x)$$

has the following properties :

(1) F *is of bounded variation on* $[a, b]$*.*

(2) *Every continuous point of* α *is also a continuous point of* F*.*