Appendix C

Useful Definite Integrals

Definite integrals that often arise in plasma physics are summarized in this appendix.

C.1 Integrals Involving A Decaying Exponential

Integrals over temporally or spatially decaying processes (e.g., collisional damping at rate $\nu \equiv 1/\tau$) often result in integrals of the form

$$\int_0^\infty dt\, t^n e^{-t/\tau} \; = \; \tau^{n+1} \int_0^\infty dx\, x^n\, e^{-x} \tag{C.1}$$

in which $x \equiv t/\tau$. The most general definite, dimensionless integral involving powers of a variable x and the exponential e^{-x} is that given by the gamma (factorial) function, which is defined by Euler's integral:

$$\Gamma(z) \; \equiv \int_0^\infty dx\, x^{z-1}\, e^{-x}, \quad \text{for } \mathcal{R}e(z) > 0. \tag{C.2}$$

Integrating by parts, one obtains the important recursion relation

$$\Gamma(z+1) \; = \; z\,\Gamma(z). \tag{C.3}$$

Using this relation recursively, the gamma function for any argument $z > 1$ can be evaluated in terms of $\Gamma(z)$ for $0 < z \leq 1$.

Two values of the argument z of fundamental interest for gamma functions are $z = 1$ and $z = 1/2$. For $z = 1$ the gamma function becomes simply the integral of a decaying exponential:

$$\Gamma(1) \; = \int_0^\infty dx\, e^{-x} \; = \; 1. \tag{C.4}$$

For $z = 1/2$, using the substitution $x = u^2$ the gamma function becomes the integral of a Gaussian distribution over an infinite domain:

$$\Gamma(\tfrac{1}{2}) \; = \; 2\int_0^\infty du\, e^{-u^2} \; = \; \sqrt{\pi}. \tag{C.5}$$

When the argument of the gamma function is a positive integer ($z \to n > 0$), the gamma function simplifies to a factorial function:

$$\Gamma(n+1) \;=\; n\,\Gamma(n) \;=\; n(n-1)\,\Gamma(n-1) \;=\; n(n-1)(n-2)\cdots 1 \;\equiv\; n!. \quad \text{(C.6)}$$

Using this factorial form for the gamma function, one thus finds that

$$\int_0^\infty dt\, t^n\, e^{-t/\tau} = \tau^{n+1}\, n!, \quad \text{for } n = 0, 1, 2, \cdots, \qquad \text{(C.7)}$$

using the usual convention that $0! \equiv 1$. The first few of these integrals are

$$\int_0^\infty \frac{dt}{\tau} \left\{ \begin{array}{r} 1 \\ t/\tau \\ t^2/\tau^2 \end{array} \right\} e^{-t/\tau} = \int_0^\infty dx \left\{ \begin{array}{c} 1 \\ x \\ x^2 \end{array} \right\} e^{-x} \;=\; \left\{ \begin{array}{c} 1 \\ 1 \\ 2 \end{array} \right\}. \qquad \text{(C.8)}$$

When the argument of the gamma function is a positive half integer ($z \to n + 1/2 > 0$), the gamma function simplifies to a double factorial:

$$\begin{aligned} \Gamma(n+\tfrac{1}{2}) \;&=\; (n-\tfrac{1}{2})\,\Gamma(n-\tfrac{1}{2}) \;=\; (n-\tfrac{1}{2})(n-\tfrac{3}{2})\,\Gamma(n-\tfrac{3}{2}) \\ &=\; [(2n-1)(2n-3)\cdots 1]\;\Gamma(\tfrac{1}{2})\,/\,2^n \;\equiv\; (2n-1)!!\,\sqrt{\pi}\,/\,2^n. \end{aligned} \qquad \text{(C.9)}$$

C.2 Integrals Over A Maxwellian

When calculating various averages over a Maxwellian distribution, integrals of the following type occur:

$$I_m = \int_0^\infty dv\, v^m\, e^{-v^2/v_T^2} \;=\; v_T^{m+1} \int_0^\infty du\, u^m\, e^{-u^2} \qquad \text{(C.10)}$$

in which m is a nonnegative integer and in the second, dimensionless integral $u \equiv v/v_T$. This integral can be calculated for arbitrary $m \geq 0$ by changing the variable of integration from u to $x = u^2 = v^2/v_T^2$ and relating the resulting integral to the gamma function, (C.2):

$$I_m \;=\; \frac{v_T^{m+1}}{2} \int_0^\infty dx\, x^{m/2-1/2}\, e^{-x} \;=\; \frac{v_T^{m+1}}{2}\, \Gamma[(m+1)/2]. \qquad \text{(C.11)}$$

The integrals for the first few even m [for which $(m+1)/2$ becomes a half integer and (C.9) applies] are

$$\int_0^\infty \frac{dv}{v_T} \left\{ \begin{array}{r} 1 \\ v^2/v_T^2 \\ v^4/v_T^4 \end{array} \right\} e^{-v^2/v_T^2} = \int_0^\infty du \left\{ \begin{array}{c} 1 \\ u^2 \\ u^4 \end{array} \right\} e^{-u^2} \;=\; \frac{\sqrt{\pi}}{2} \left\{ \begin{array}{c} 1 \\ 1/2 \\ 3/4 \end{array} \right\}. \qquad \text{(C.12)}$$

The integrals for the first few odd m [for which $(m+1)/2$ becomes an integer and (C.6) applies] are

$$\int_0^\infty \frac{dv}{v_T} \left\{ \begin{array}{c} v/v_T \\ v^3/v_T^3 \\ v^5/v_T^5 \end{array} \right\} e^{-v^2/v_T^2} = \int_0^\infty du \left\{ \begin{array}{c} u \\ u^3 \\ u^5 \end{array} \right\} e^{-u^2} = \left\{ \begin{array}{c} 1/2 \\ 1/2 \\ 1 \end{array} \right\}. \qquad \text{(C.13)}$$

The natural (orthogonal basis) energy weighting functions for expanding distribution functions in terms of fluid moments are the Laguerre polynomials $L_n^{l+1/2}(x)$, which are defined and discussed in Section B.6. The relevant dimensionless integral of products of Laguerre polynomials that indicates their orthogonality and normalization is

$$\begin{aligned} &\int_0^\infty dx\, x^{l+1/2}\, e^{-x}\, L_n^{(l+1/2)}(x)\, L_{n'}^{(l+1/2)}(x) \\ &= \delta_{nn'} \frac{\Gamma(l+n+3/2)}{\Gamma(n+1)} = \delta_{nn'} \left[2(n+l)+1\right]!! \frac{\sqrt{\pi}}{2^{n+l+1}\, n!} \end{aligned} \qquad \text{(C.14)}$$

in which $x \equiv v^2/v_T^2 = mv^2/2T$, and $\delta_{nn'}$ is the Kronecker delta, which is unity for $n = n'$ and vanishes if $n \neq n'$. The lowest order ($n = 0, 1, 2$ and $l = 0, 1, 2$) integrals of interest are

$$\begin{aligned} &\int_0^\infty dx\, x^{1/2} e^{-x} \left(\begin{array}{ccc} [L_0^{(1/2)}]^2 & [L_1^{1/2}]^2 & [L_2^{1/2}]^2 \\ x[L_0^{(3/2)}]^2 & x[L_1^{(3/2)}]^2 & x[L_2^{(3/2)}]^2 \\ x^2[L_0^{(5/2)}]^2 & x^2[L_1^{(5/2)}]^2 & x^2[L_2^{(5/2)}]^2 \end{array} \right) \\ &= \frac{\sqrt{\pi}}{2} \left(\begin{array}{ccc} 1 & 3/2 & 15/4 \\ 3/2 & 15/4 & 105/8 \\ 15/8 & 105/16 & 945/32 \end{array} \right). \end{aligned} \qquad \text{(C.15)}$$

C.3 Integrals Over Sinusoidal Functions

Averaging linear and nonlinear quantities made up of sinusoidally oscillating components result in integrals of the form

$$\langle \sin^m\varphi\ \cos^n\varphi \rangle_\varphi \equiv \frac{1}{2\pi} \int_0^{2\pi} d\varphi\, \sin^m\varphi\ \cos^n\varphi. \qquad \text{(C.16)}$$

Trigonometric identities that are useful in reducing these integrals to simpler forms are

$$2\sin\varphi\cos\varphi = \sin 2\varphi, \qquad \text{(C.17)}$$
$$2\sin^2\varphi = (1-\cos 2\varphi), \qquad \text{(C.18)}$$
$$2\cos^2\varphi = (1+\cos 2\varphi), \qquad \text{(C.19)}$$

which are derivable from the more fundamental trigonometric identities

$$\sin(\varphi_1 + \varphi_2) = \sin\varphi_1 \cos\varphi_2 + \cos\varphi_1 \sin\varphi_2, \tag{C.20}$$
$$\cos(\varphi_1 + \varphi_2) = \cos\varphi_1 \cos\varphi_2 - \sin\varphi_1 \sin\varphi_2. \tag{C.21}$$

These last two identities can be also combined to yield

$$2\sin\varphi_1 \sin\varphi_2 = \cos(\varphi_1 + \varphi_2) - \cos(\varphi_1 - \varphi_2), \tag{C.22}$$
$$2\sin\varphi_1 \cos\varphi_2 = \sin(\varphi_1 + \varphi_2) + \sin(\varphi_1 - \varphi_2), \tag{C.23}$$
$$2\cos\varphi_1 \cos\varphi_2 = \cos(\varphi_1 + \varphi_2) + \cos(\varphi_1 - \varphi_2). \tag{C.24}$$

Using these trigonometric identities, and the facts that $\int_0^{2\pi} d\varphi \sin n\varphi = 0$ and $\int_0^{2\pi} d\varphi \cos n\varphi = 0$ for $n = 1, 2, \cdots$, it can be shown that

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi \begin{pmatrix} 1 & \sin\varphi & \cos\varphi \\ \sin\varphi\cos\varphi & \sin^2\varphi & \cos^2\varphi \\ \sin\varphi\cos^2\varphi & \sin^3\varphi & \cos^3\varphi \\ \sin^2\varphi\cos^2\varphi & \sin^4\varphi & \cos^4\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 1/8 & 3/8 & 3/8 \end{pmatrix}. \tag{C.25}$$

The natural (i.e., orthogonal basis) functions of sinusoidal functions in which to expand spherical velocity space latitude angle dependences are the Legendre polynomials $P_l(\zeta)$, which are defined and discussed in Section B.5. The relevant argument of the Legendre polynomials is usually $\zeta \equiv \cos\vartheta$. The relevant integral of products of Legendre polynomials that indicates their orthogonality and normalization is

$$\int_{-1}^{1} d\zeta\, P_l(\zeta)\, P_{l'}(\zeta) = \int_0^{\pi} d\vartheta\, \sin\vartheta\, P_l(\cos\vartheta)\, P_{l'}(\cos\vartheta) = \frac{2\,\delta_{ll'}}{2l+1} \tag{C.26}$$

in which $\delta_{ll'}$ is the Kronecker delta function which is unity if the indices are equal and zero otherwise. The first few of these nonvanishing integrals are

$$\int_{-1}^{1} d\zeta \begin{pmatrix} P_0^2 \\ P_1^2 \\ P_2^2 \end{pmatrix} \equiv \int_{-1}^{1} d(\cos\vartheta) \begin{pmatrix} 1 \\ \cos^2\vartheta \\ (3\cos^2\vartheta - 1)^2/4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2/3 \\ 2/5 \end{pmatrix}. \tag{C.27}$$

REFERENCES

A limited but very useful table of integrals is:

Dwight, *Tables of Integrals and Other Mathematical Data* (1964) [**?**]

The most comprehensive tabulation of integrals is provided by:

Gradshteyn and Ryzhik, *Table of Integrals, Series and Products* (1965) [**?**]