Bayesian inference in Inverse problems

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Inverse Problems

- Inverse problems arise from indirect observations of a quantity of interest
- Observations may be limited in numbers relative to the dimension or complexity of the model space
- Inverse problems ill posed
- Classical approaches have used regularization methods to impose well-posedness ans solved the resulting deterministic problems by optimization

Bayesian approach

- A natural mechanism for regularization in the form of prior information
- Can handle non linearity, non Gaussianity
- Focus is on uncertainties in parameters, as much as on their best (estimated) value.
- Permits use of prior knowledge, e.g., previous experiments, modeling expertise, physics constraints.
- Model-based.
- Can add data sequentially

Inverse problem

- Inverse problems whose solutions are unknown functions: Spatial or temporal fields
- Estimating fields rather than parameters typically increases the ill-posedness of the inverse problem since one is recovering an infinite dimensional object from finite amounts of data
- Obtaining physically meaningful results requires the injection of additional information on the unknown field
- A standard Bayesian approach is to employ Gaussian process or Markov Random field priors

Forward Model and Inverse problem

$$Z = F(K) + \epsilon$$

where

- F is the forward model, simulator, computer code which is non-linear and expensive to run.
- K is a spatial field
- Z is the observed response
- ϵ is the random error usually assumed to be Gaussian
- Want to estimate K with UQ
- This is a non-linear inverse problem

Fluid flow in porous media

- Studying flow of liquids (Ground water, oil) in aquifer (reservoir)
- Applications: Oil production, Contaminant cleanup
- Forward Model: Models the flow of liquid, output is the production data, inputs are physical characteristics like permeability, porosity
- Inverse problem: Inferring the permeability from the flow data

Permeability

- Primary parameter of interest is the permeability field
- Permeability is a measure of how easily liquid flows through the aquifer at that point
- This permeability values vary over space
- Effective recovery procedures rely on good permeability estimates, as one must be able to identify high permeability channels and low permeability barriers

Forward Model

Darcy's law:

$$v_j = -\frac{k_{rj}(S)}{\mu_j} k_f \nabla p, \tag{1}$$

- v_j is the phase velocity
- k_f is the fine-scale permeability field
- k_{rj} is the relative permeability to phase j (j=oil or water)
- S is the water saturation (volume fraction)
- p is the pressure.

Forward Model

Combining Darcy's law with a statement of conservation of mass allows us to express the governing equations in terms of pressure and saturation equations:

$$\nabla \cdot (\lambda(S) k_f \nabla p) = Q_s, \tag{2}$$

$$\frac{\partial S}{\partial t} + v \cdot \nabla f(S) = 0, \tag{3}$$

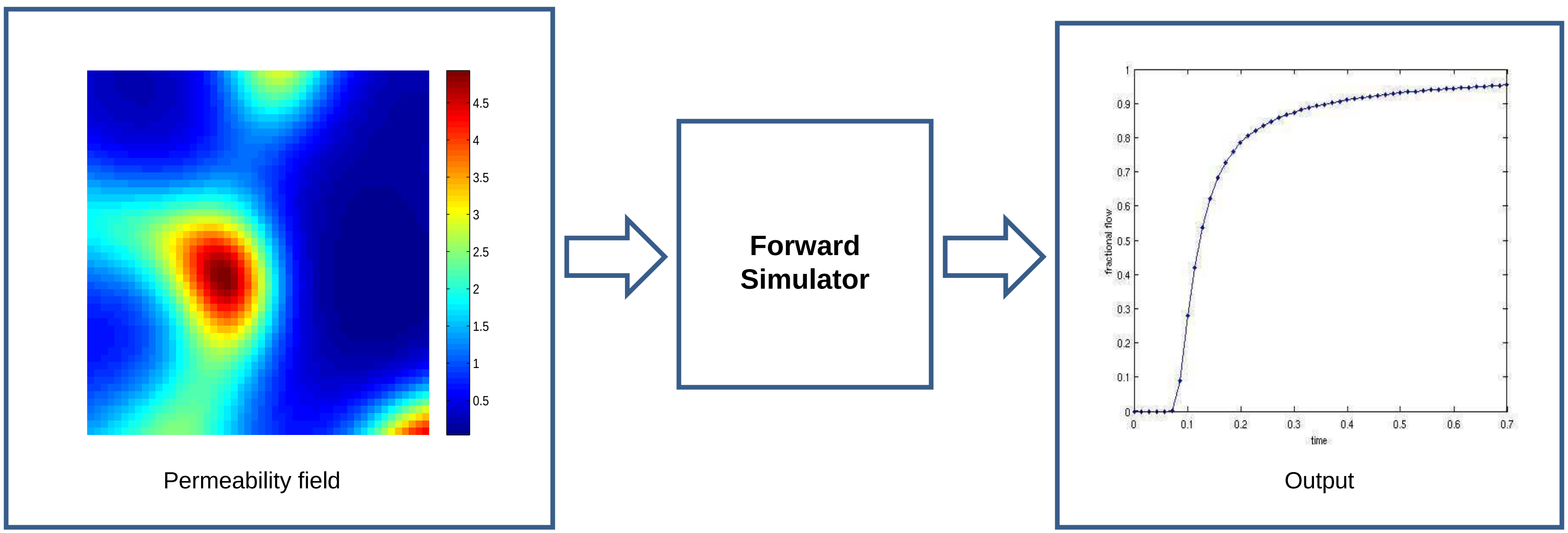
- λ is the total mobility
- Q_s is a source term
- f is the fractional flux of water
- v is the total velocity

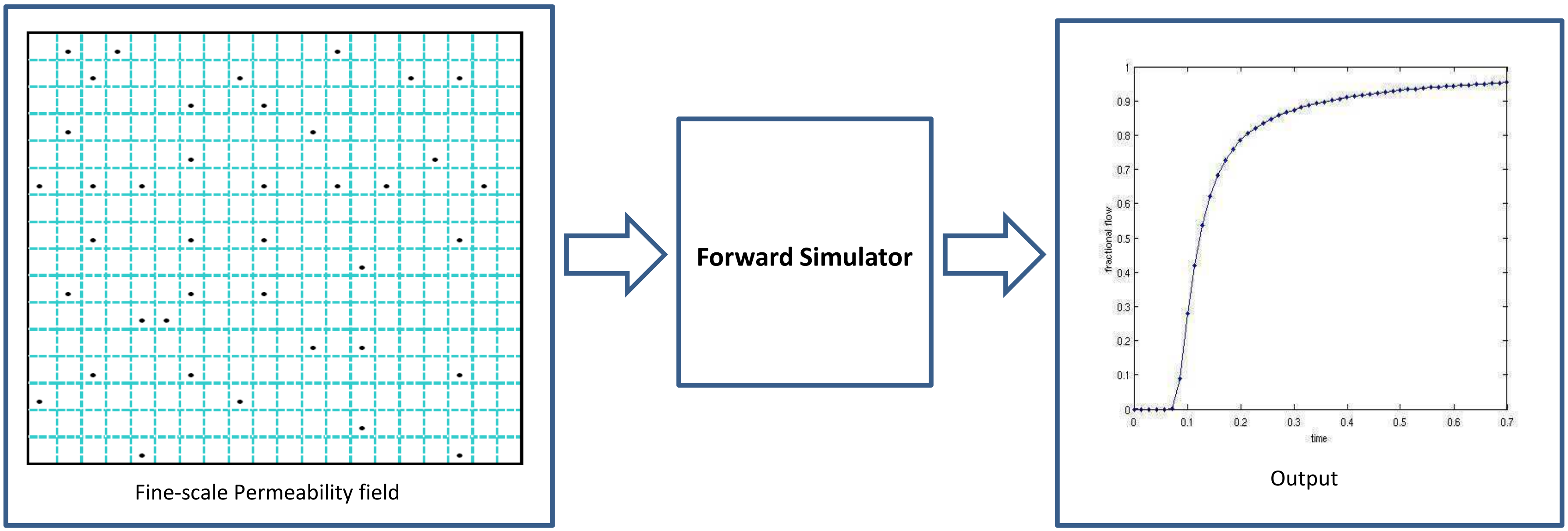
Forward Model

Production (amount of oil in the produced fluid, fractional Flow or water-cut) $F_{k_f}(t)$ is given by

$$F_{k_f}(t) = \int_{\partial\Omega^{out}} v_n f(S) dl \qquad (4)$$

where $\partial\Omega^{out}$ is outflow boundaries and v_n is normal velocity field.


4.5
4
3.5
3
2.5
2
1.5
1
0.5
Permeability field
Forward Simulator
fractional flow
time
Output


Fine-scale Permeability field
Forward Simulator
fractional flow
time
Output

Bayesian way

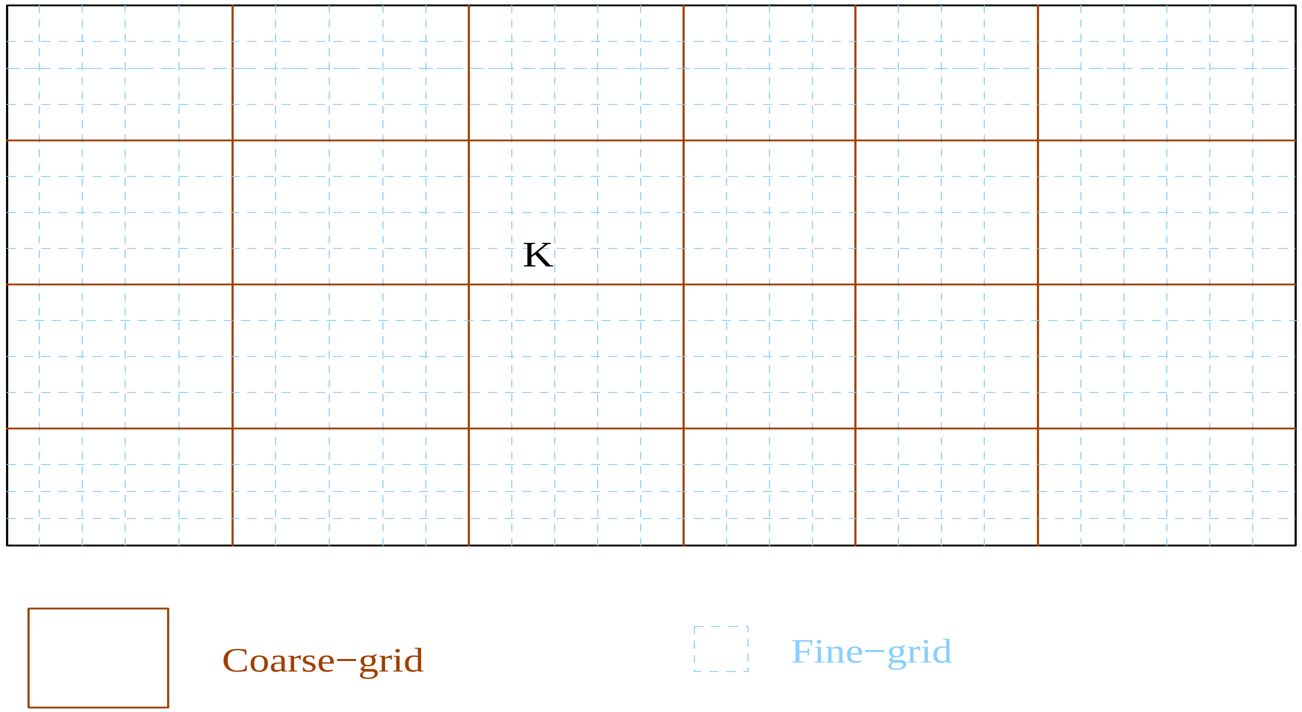
- If $p(K)$ is the prior for the spatial field K: usually Gaussian processes
- $p(Z|k)$ is the likelihood depending on the distribution of ϵ: Gaussian, non-Gaussian
- Then posterior distribution: $p(K|Z) \propto p(Z|K)p(K)$ is the Bayesian solution of this inverse problem

Inverse Problem

- Dimension reduction:Replacing K by a finite set of parameters τ
- Building enough structures through models and priors.
- Additional data: coarse-scale data
- Need to link data at different scales
- Bayesian hierarchical models have the ability to do all these things simultaneously

Multiscale Data

- K_f is the fine scale field of interest (data: well logs, cores)
- Additional data: from coarse scale field K_c (seismic traces)
- Some of the observed fine-scale permeability values K_f^o at some spatial locations
- We want to infer K_f conditioned on Z, K_c and K_f^o
- The posterior distribution of interest: $p(K_f|Z, K_c, K_f^o)$



No flow

$$\phi=1 \qquad \boxed{div(k_f(x)\Delta\phi)=0} \qquad \phi=0$$

No flow

$$(k_c(x)e_j,e_l)=\frac{1}{|K|}\int_K (k_f(x)\Delta\phi_j(x),e_l)dx$$

Dimension reduction

- We need to reduce the dimension of the spatial field K_f
- This is a spatial field denoted by $K_f(\boldsymbol{x}, \omega)$ where $\boldsymbol{x}$ is for the spatial locations and ω denotes the randomness in the process
- Assuming K_f to be a real-valued random field with finite second moments we can represent it by Kauren-Loeve (K-L) expansion

K-L expansion

$$K_f(\boldsymbol{x}, \omega) = \theta_0 + \sum_{l=1}^{\infty} \sqrt{\lambda_l}\theta_l(\omega)\phi_l(\boldsymbol{x})$$

where

- λ: eigen values
- $\phi(\boldsymbol{x})$ eigen functions
- θ: uncorrelated with zero mean and unit variance
- If K_f is Gaussian process then θ will be Gaussian

K-L expansion

If the covariance kernel is C then we obtain them by solving

$$\int C(\boldsymbol{x}_1, \boldsymbol{x}_2)\phi_l(\boldsymbol{x}_2)d\boldsymbol{x}_2 = \lambda_l \phi_l(\boldsymbol{x}_1)$$

and can express C as

$$C(\boldsymbol{x}_1, \boldsymbol{x}_2) = \sum_{l=1}^{\infty} \lambda_l \phi_l(\boldsymbol{x}_1)\phi_l(\boldsymbol{x}_2)$$

Spatial covariance

We assume the correlation structure

$$C(\mathbf{x},\mathbf{y}) = \sigma^2 \exp\left(-\frac{|x_1-y_1|^2}{2l_1^2} - \frac{|x_2-y_2|^2}{2l_2^2}\right).$$

where, l_1 and l_2 are correlation lengths.

For an m-term KLE approximation

$$\begin{aligned} K_f^m &= \theta_0 + \sum_{i=1}^{m} \sqrt{\lambda_i}\theta_i \Phi_i, \\ &= B(l_1, l_2, \sigma^2)\theta, \text{(say)} \end{aligned} \qquad (1)$$

Existing methods

- The energy ratio of the approximation is given by

$$e(m) := \frac{E\|k_f^m\|^2}{E\|k_f\|^2} = \frac{\sum_{i=1}^{m} \lambda_i}{\sum_{i=1}^{\infty} \lambda_i}.$$

- Assume correlation length l_1, l_2 and σ^2 are known.
- We treat all of them as model parameters, hence $\tau = (\theta, \sigma^2, l_1, l_2, m)$.

Hierarchical Bayes' model

$$P(\theta, l_1, l_2, \sigma^2 | Z, k_c, k_f^o) \;\propto\; P(z|\theta, l_1, l_2, \sigma^2) P(k_c|\theta, l_1, l_2, \sigma^2)$$
$$P(k_f^o|\theta, l_1, l_2, \sigma^2) P(\theta) P(l_1, l_2) P(\sigma^2)$$

- $P(z|\theta, l_1, l_2, \sigma^2)$: Likelihood
- $P(k_c|\theta, l_1, l_2, \sigma^2)$: Upscale model linking fine and coarse scales
- $P(k_f^o|\theta, l_1, l_2, \sigma^2)$: Observed fine scale model
- $P(\theta)P(l_1, l_2)P(\sigma^2)$: Priors

Likelihood

The likelihood can be written as follows:

$$
\begin{aligned}
Z &= F[B(l_1, l_2, \sigma^2)\theta] + \epsilon_f \\
&= F_1(\theta, l_1, l_2, \sigma^2) + \epsilon_f
\end{aligned}
$$

where, $\epsilon_f \sim MVN(0, \sigma_f^2 I)$.

Likelihood calculations

$$Z = F(\tau) + \epsilon$$

For Gaussian model the likelihood will be

$$P(Z|\tau) = \frac{1}{\sqrt{2\pi}\sigma_1} Exp(\frac{-[Z - F(\tau)]^2}{2\sigma_1^2})$$

where σ_1^2 is the variance of ϵ.

Likelihood Calculations

- It is like a black-box likelihood which we can't write analytically, although we do have a code F that will compute it.
- We need to run F to compute the likelihood which is expensive.
- Hence, no hope of having any conjugacy in the model, other than for the error variance in the likelihood.
- Need to be somewhat intelligent about the update steps during MCMC so that do not spend too much time computing likelihoods for poor candidates.

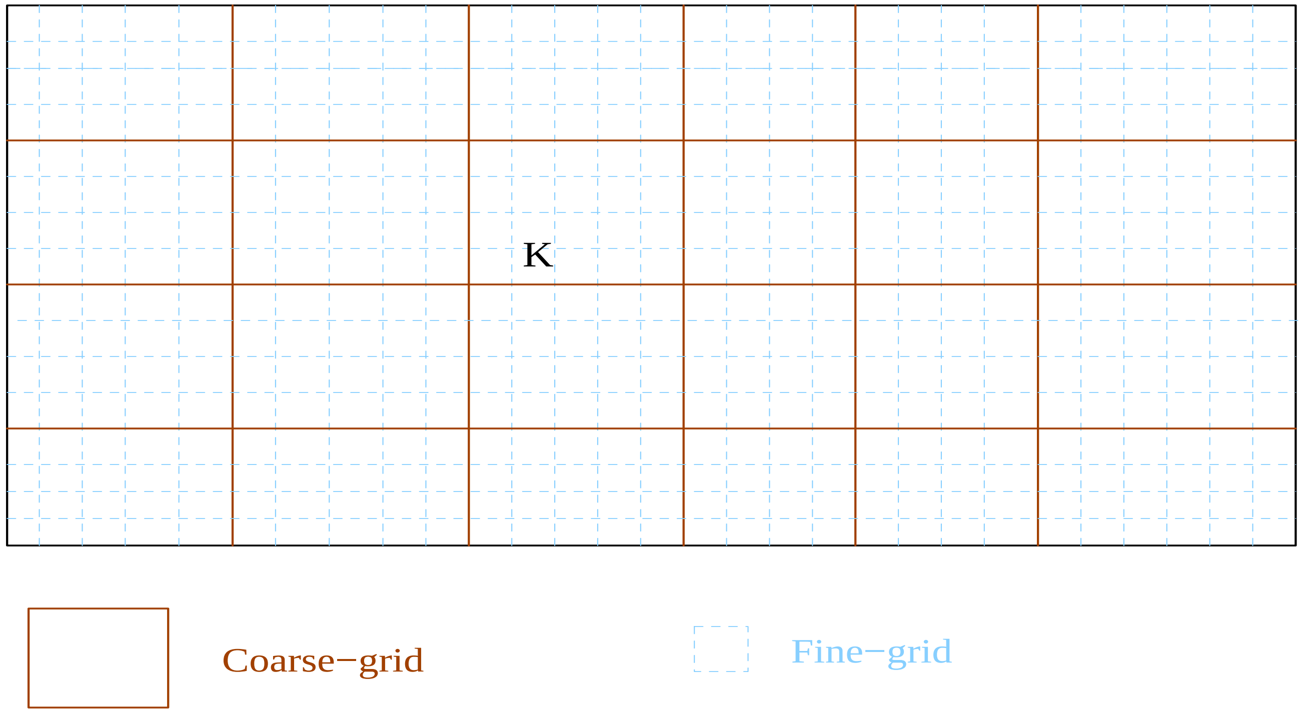
Upscale model

The Coarse-scale model can be written as follows.

$$\begin{aligned} k_c &= L_1(k_f) + \epsilon_c \\ &= L_1(\theta, l_1, l_2, \sigma^2) + \epsilon_c \end{aligned}$$

where, $\epsilon_c \sim MVN(0, \sigma_c^2 I)$.
i.e $k_c | \theta, l_1, l_2, \sigma^2, \sigma_c^2 \sim MVN(L_1(\theta, l_1, l_2, \sigma^2), \sigma_c^2 I)$.

- L_1 is the upsacling operator
- It could be as simple as average
- It could be more complex where you need to solve the original system on the coarse grid with boundary conditions



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No flow

$$(k_c(x)e_j, e_l) = \frac{1}{|K|}\int_K (k_f(x)\Delta\phi_j(x), e_l)\,dx$$

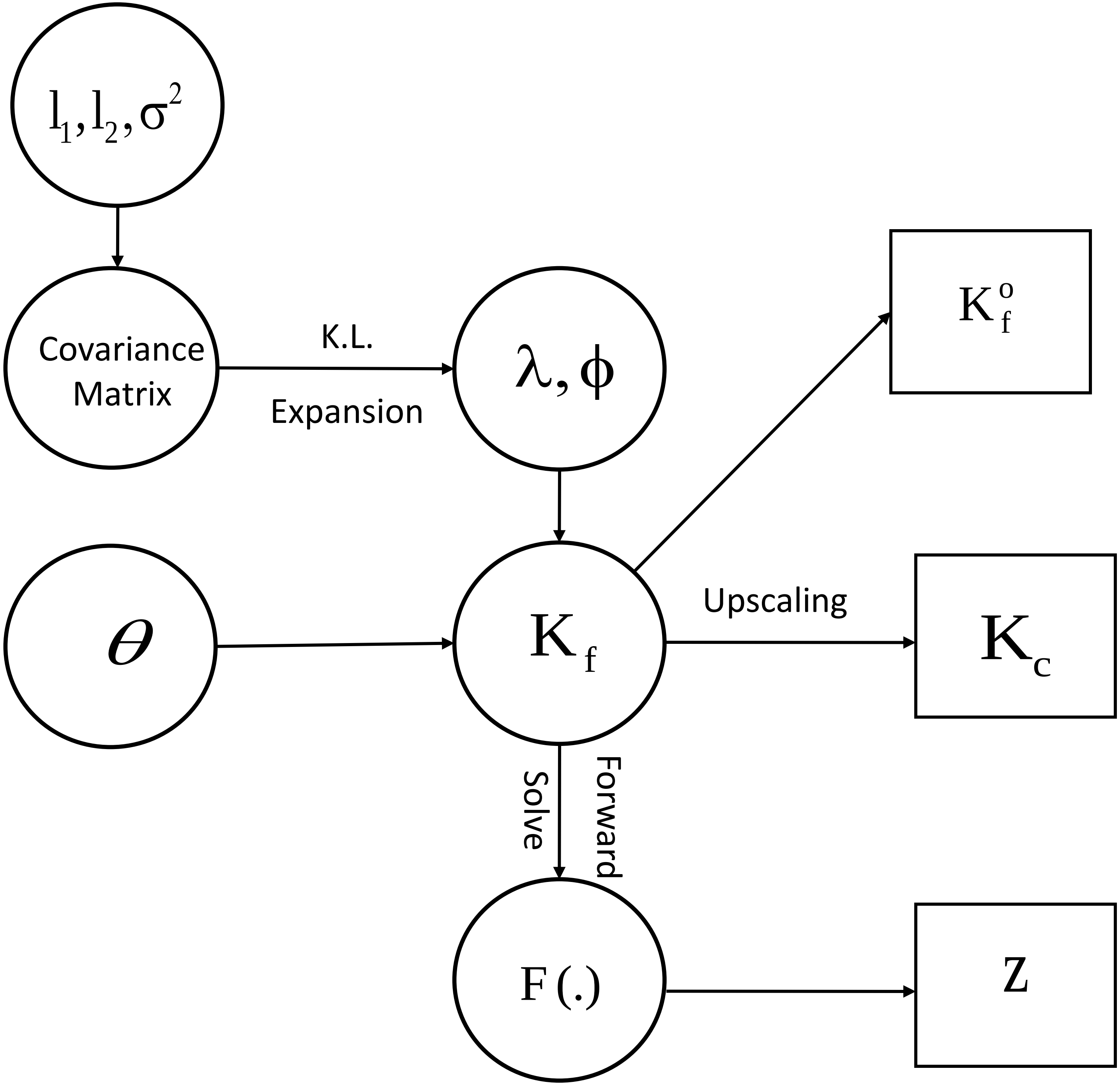
Observed fine scale model

We assume the model $k_f^o = k_p^o + \epsilon_k$

where, $\epsilon_k \sim MVN(0, \sigma_k^2)$.

k_p^o is the spatial field obtained from K-L the expansion at the observed well locations.

So here we assume, $k_f^o | \theta, l_1, l_2, \sigma^2, \sigma_k^2 \sim MVN(k_p^o, \sigma_k^2)$,



l_1, l_2, σ^2
Covariance Matrix
K.L.
Expansion
λ, ϕ
K_f^o
θ
K_f
Upscaling
K_c
Solve
Forward
F(.)
Z

Inverse problem

- We can show that the posterior measure is Lipschitz continuous with respect to the data in the total variation distance
- It guaranties that this Bayesian inverse problem is well-posed
- Say, y is the total dataset, i.e, $y = \begin{pmatrix} z \\ k_c \\ k_f^0 \end{pmatrix}$
- $g(\tau, y)$ is the likelihood and $\pi_0(\tau)$ is the prior

Inverse problem

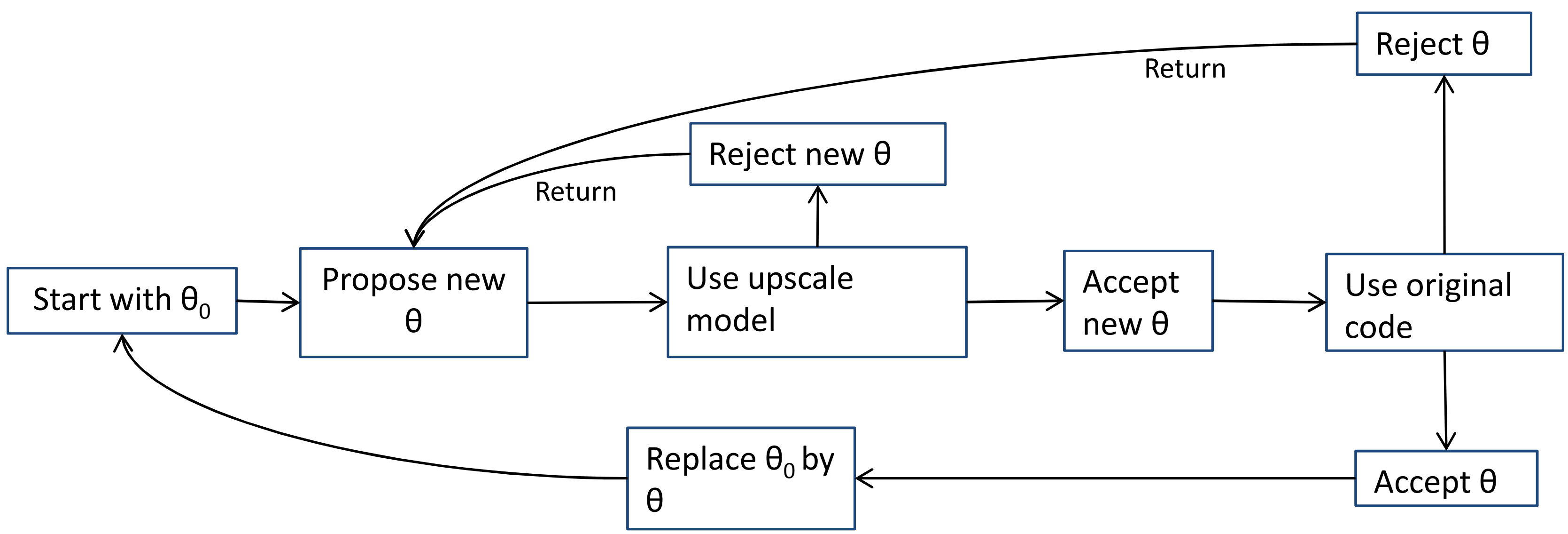
Theorem 0.1. $\forall\, r > 0,\ \exists\, C = C(r)$ *such that the posterior measures* π_1 *and* π_2 *for two different data sets* y_1 *and* y_2 *with* $max\,(\|y_1\|_{l^2}, \|y_2\|_{l^2}) \le r$*, satisfy*

$$\|\pi_1 - \pi_2\|_{TV} \le C\|y_1 - y_2\|_{l_2},$$

MCMC computation

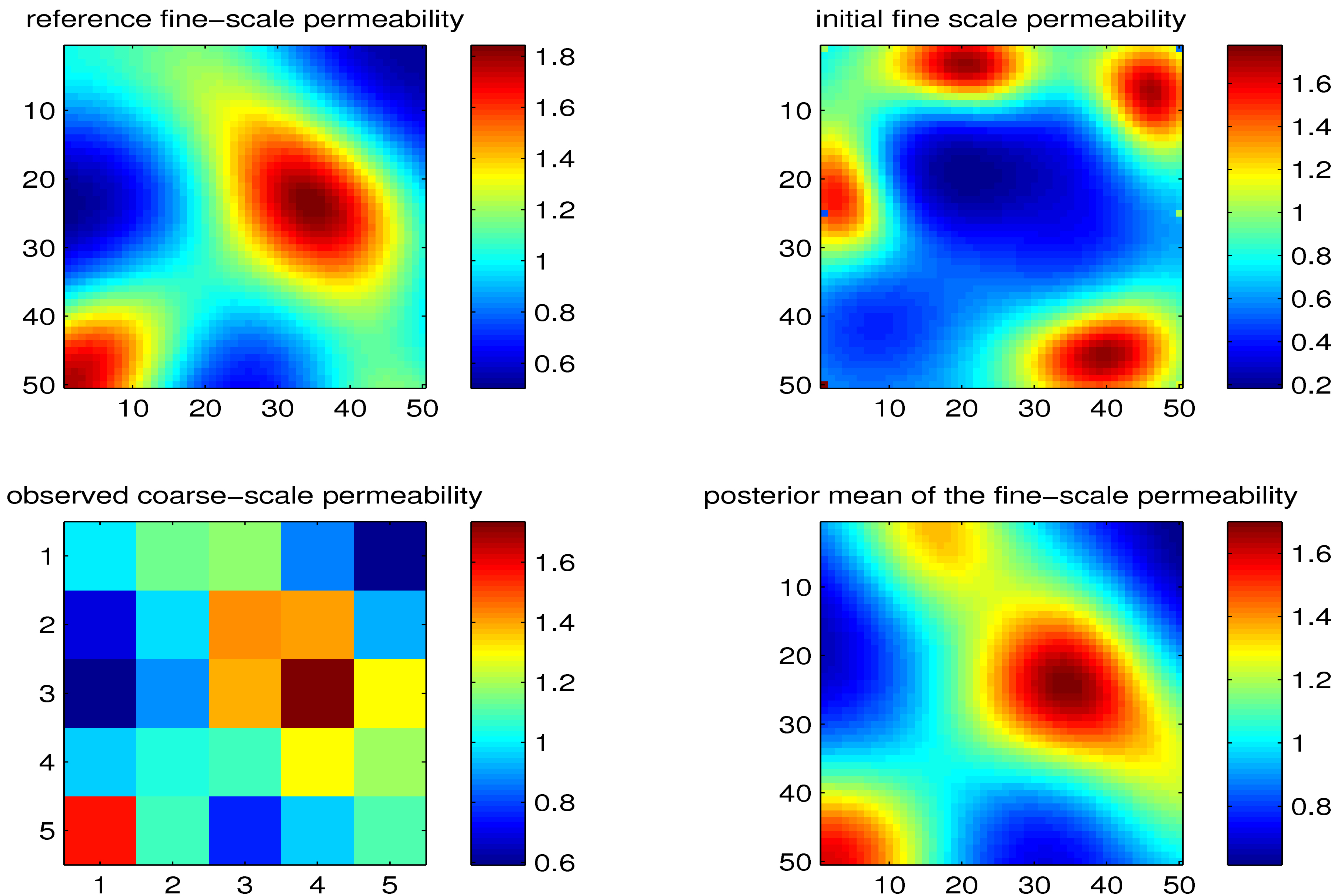
- Metropolis-Hastings (M-H) Algorithm to generate the parameters.
- Reversible jump M-H algorithm when the dimension m of the K-L expansion is treated as model unknown.
- Two step MCMC or Langevin can accelerate our computation.

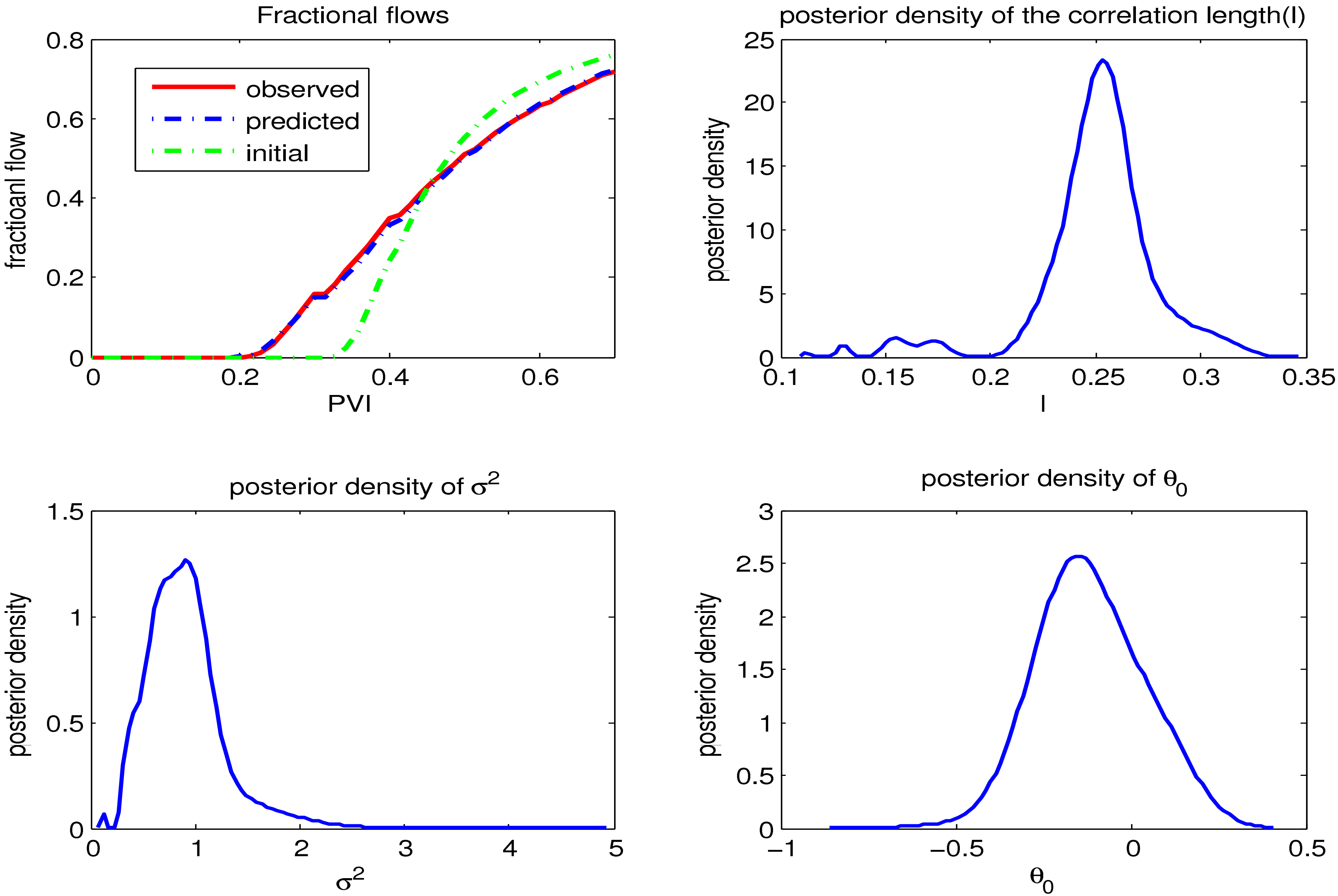
- Two stage Metropolis

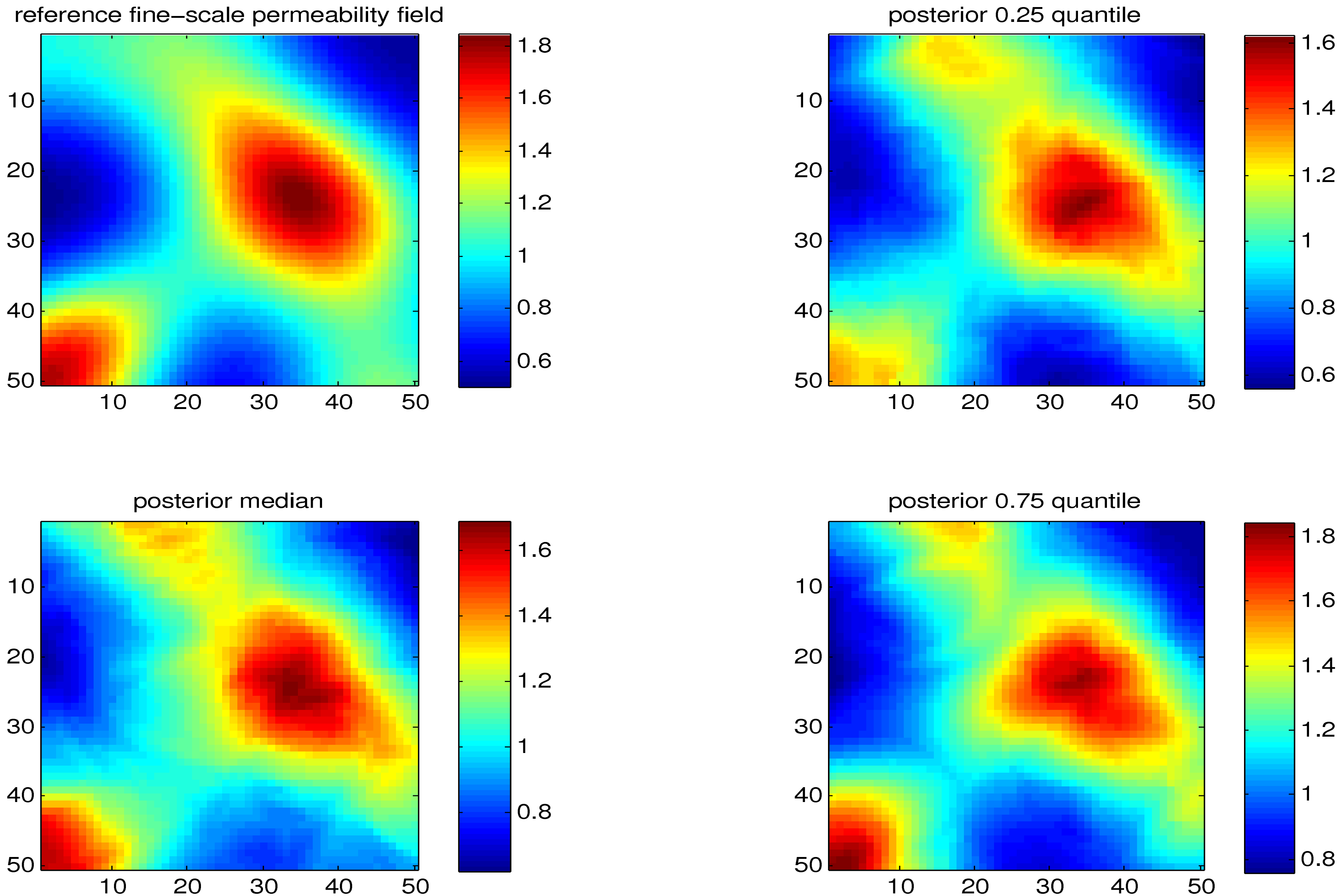


Numerical Results

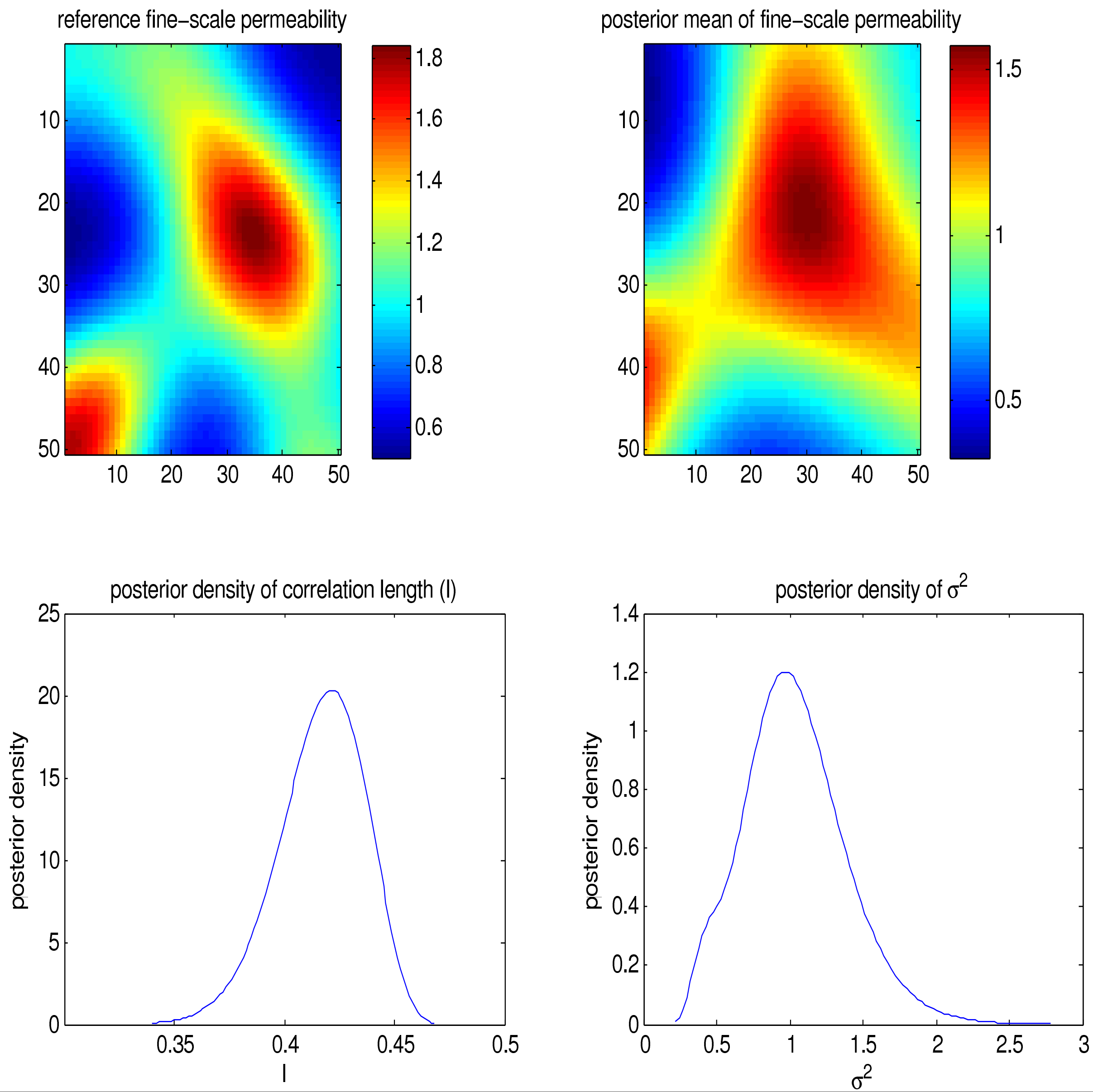
- We consider the isotropic case $l_1 = l_2 = l$
- We consider a 50X50 fine scale permeability field on unit square
- We fix $l = .25$ and $\sigma^2 = 1$
- The observed coarse-scale permeability field is calculated in a 5X5 coarse grid
- The fine-scale permeability field is observed at 100 locations


reference fine−scale permeability
initial fine scale permeability
observed coarse−scale permeability
posterior mean of the fine−scale permeability

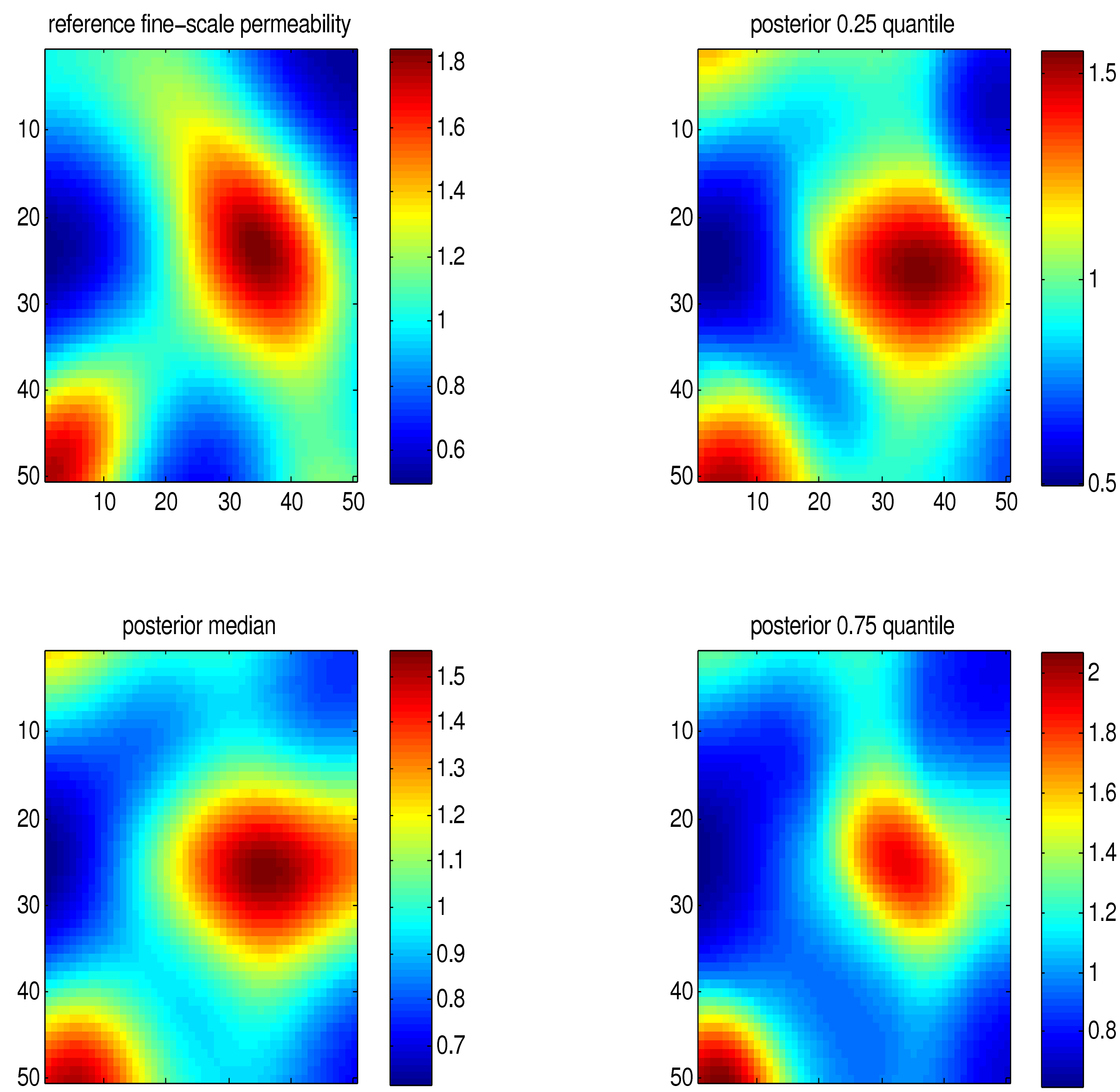

Fractional flows
fractioanl flow
PVI
observed
predicted
initial
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
posterior density of the correlation length(l)
posterior density
l
25
20
15
10
5
0
0.1
0.15
0.2
0.25
0.3
0.35
posterior density of σ^2
posterior density
σ^2
1.5
1
0.5
0
0
1
2
3
4
5
posterior density of θ_0
posterior density
θ_0
3
2.5
2
1.5
1
0.5
0
−1
−0.5
0
0.5


reference fine-scale permeability field
10
20
30
40
50
10 20 30 40 50
1.8
1.6
1.4
1.2
1
0.8
0.6
posterior 0.25 quantile
10
20
30
40
50
10 20 30 40 50
1.6
1.4
1.2
1
0.8
0.6
posterior median
10
20
30
40
50
10 20 30 40 50
1.6
1.4
1.2
1
0.8
posterior 0.75 quantile
10
20
30
40
50
10 20 30 40 50
1.8
1.6
1.4
1.2
1
0.8

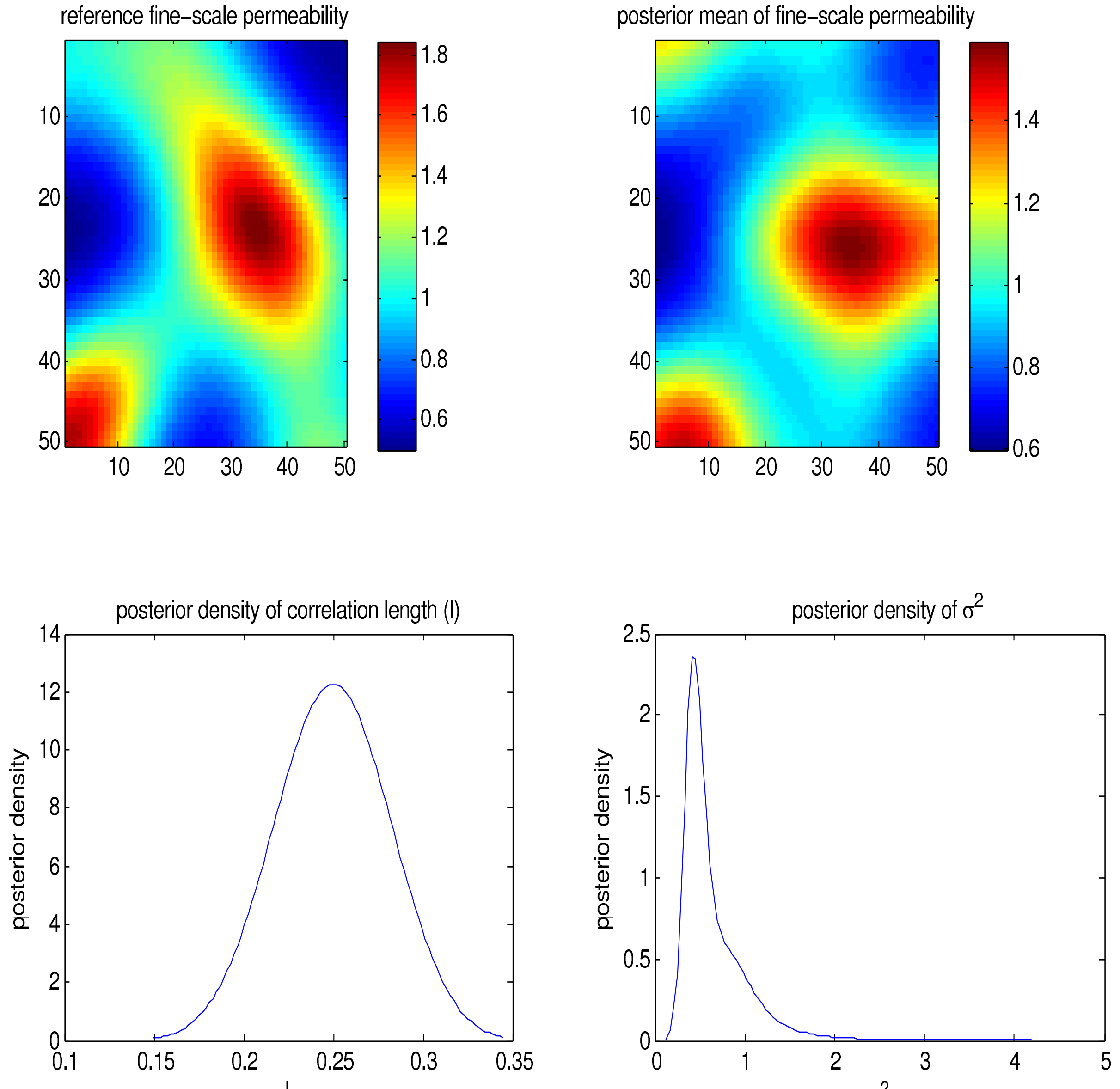
10 percent fine-scale data observed and no coarse-scale data available



25 percent fine-scale data observed and no coarse-scale data available



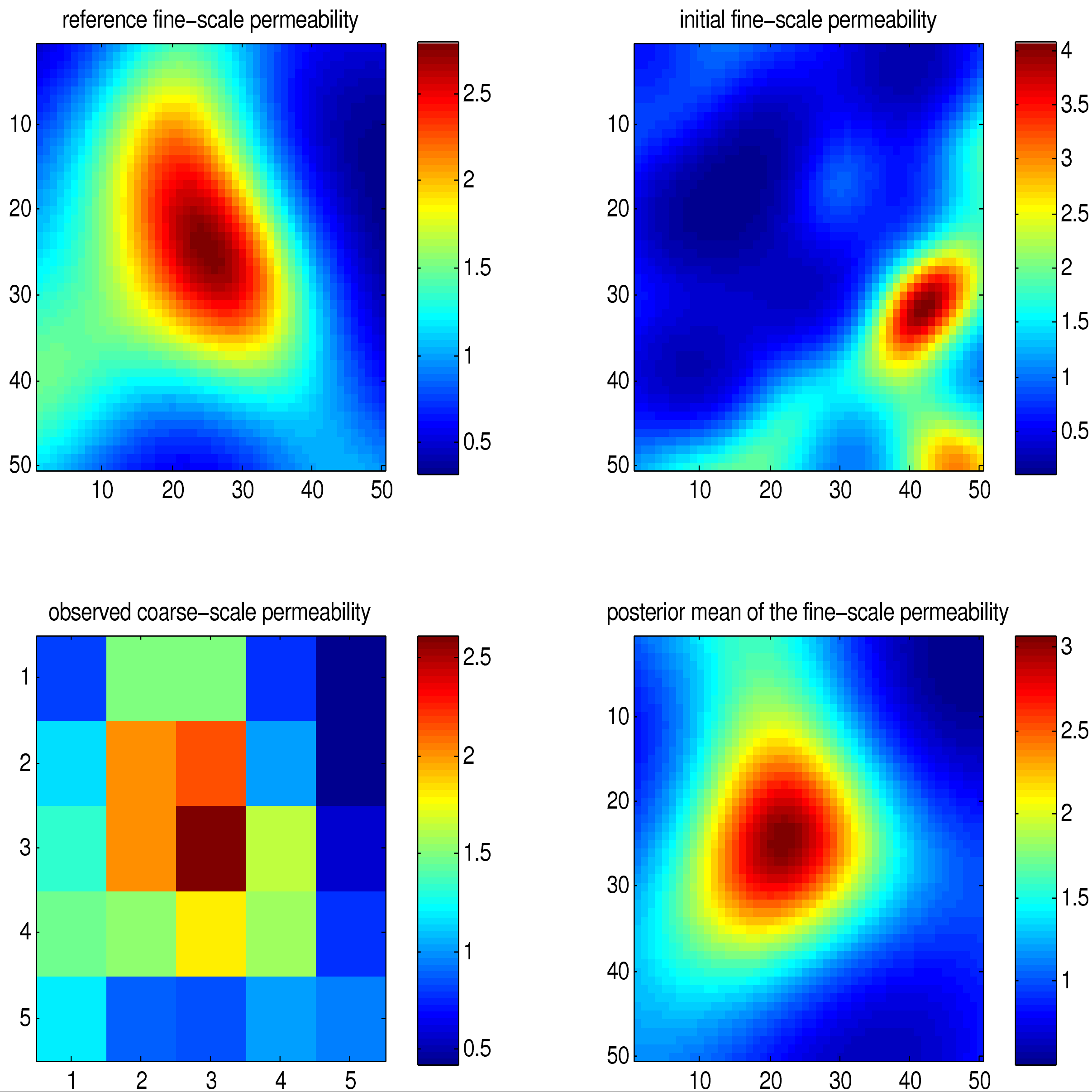
25 percent fine-scale data observed and no coarse-scale data available



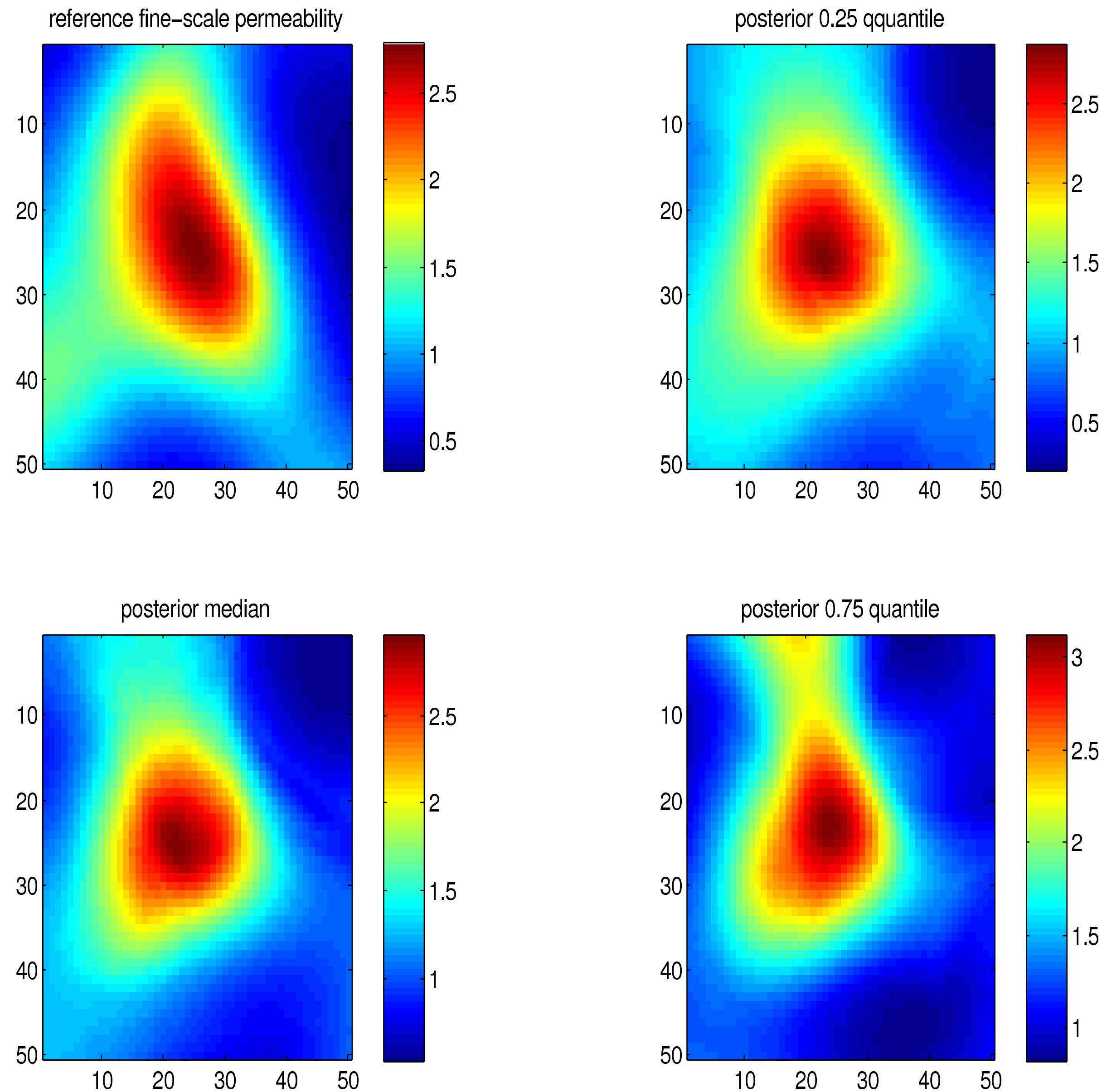
Numerical results with unknown K-L terms

- We generate 15 fine-scale permeability field with $l = .3$, $\sigma^2 = .2$ and the reference permeability field is taken to be the average of these 15 permeability field.
- We take the first 20 terms in the K-L expansion while generating the reference field.
- The mode of the posterior distribution of m comes out to be 19.
- The posterior mean of fine-scale permeability field resembles very close to the reference permeability field.
- The posterior density of l is bimodal but the highest peak is near $.3$.
- The posterior density σ^2 are centered around $.2$.

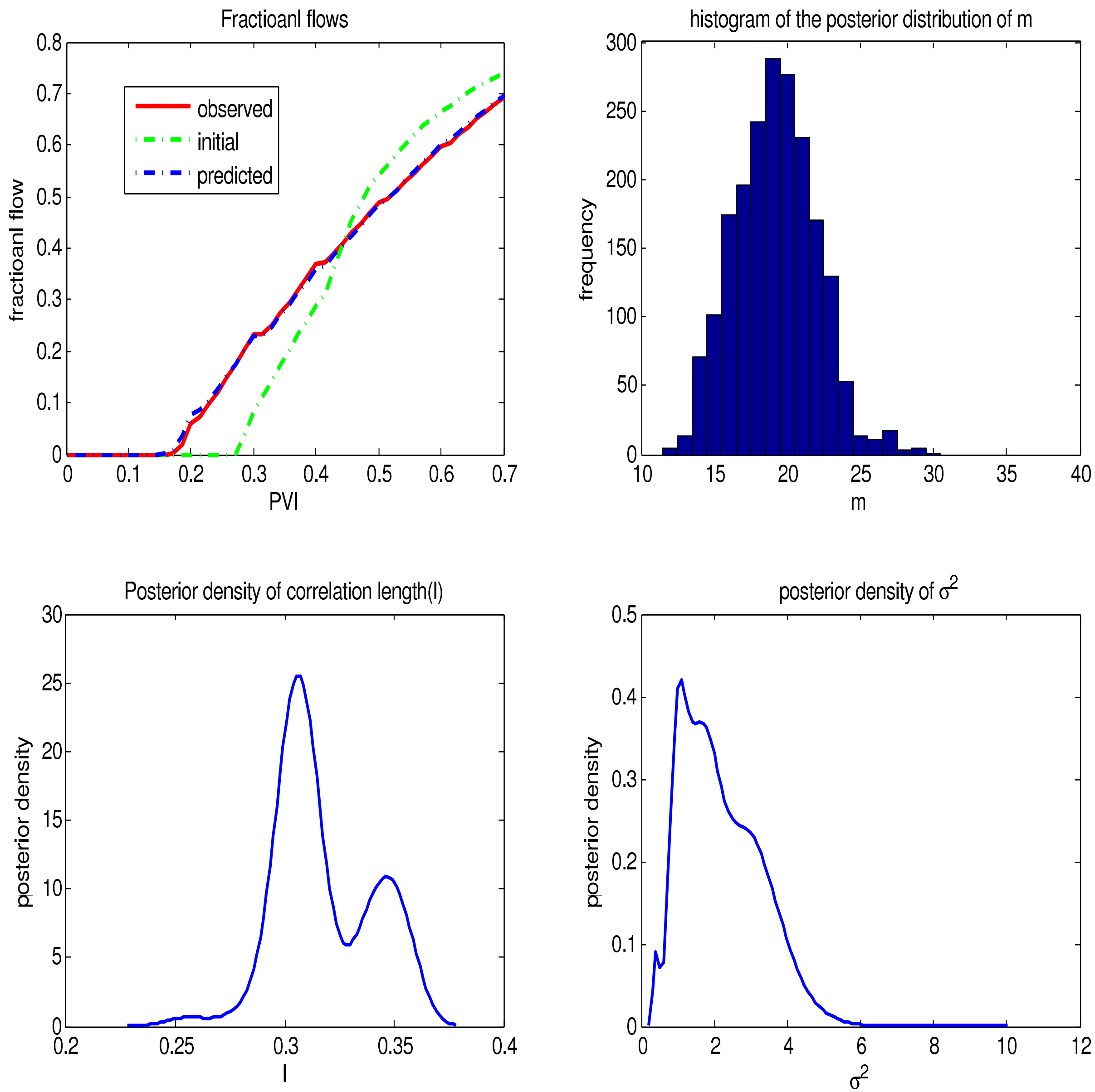
Numerical Results using Reversible Jump MCMC



Numerical Results using Reversible Jump MCMC



Numerical Results using Reversible Jump MCMC



Discussion

- We have developed a Bayesian hierarchical model which is very flexible. We can use it for other fluid dynamics, weather forecasting problems
- To use other dimension reduction techniques like predictive processes
- In two stage MCMC: can we use approximate solvers (Polynomial Chaos,...) or emulators at the the first stage
- Bayes Theorem in Infinite dimension: Warwick, A. Stuart and his group