From Branching Processes to Partial Differential Equations

José Alfredo López Mimbela
Centro de Investigación en Matemáticas
Guanajuato, Mexico

Second Pan-American Advanced Studies Institute on Biostatistics, Survival Analysis, Bioinformatics, and Statistical Genetics.

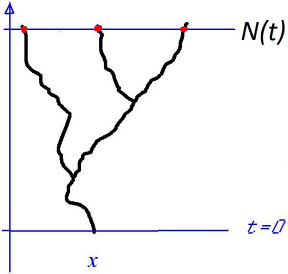
Xalapa, Veracruz – August 2011

A Branching Particle System is a branching process such that

- The time parameter is continous
- Each individual posesses a random lifetime, at the end of which it branches
- Each individual posesses a random location in certain measurable space, say $\mathbb{R}^d$.

The spatial position of individuals becomes relevant if, in addition, the individuals perform independent migrations during their lifetimes.

A heuristic pictorial description of a BPS looks like this:



Writing $\delta_x(A) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$, $A \in \mathcal{B}(\mathbb{R}^d)$, we see that

$$N_t(A) = \sum_i \delta_{x_i(t)}(A) = \text{number of particles in } A \text{ at time } t.$$

For each $t \geq 0$,

- N_t represents the state of the population at time t,
- N_t can be suitably modeled by a random point measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

What is a random point measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$?

$\mathcal{N}(\mathbb{R}^d) \quad := \quad$ space of counting meaures μ on $\mathbb{R}^d$ such that $\mu(A) < \infty$ for each compact set $A \subset \mathbb{R}^d$.

Theorem

Let $\mu \in \mathcal{N}(\mathbb{R}^d)$. For any compact $K \subset \mathbb{R}^d$, either

$$\mu(K) = 0,$$

or there exist $x_1, \ldots, x_{n_K} \in \mathbb{R}^d$ and $j_1, \ldots, j_{n_k} \in \mathbb{N}$ such that

$$\mu(A) = j_1 \delta_{x_1}(A) + \cdots + j_{n_K} \delta_{x_{n_K}}(A), \quad A \in \mathcal{B}(K).$$

For any measurable $\varphi : \mathbb{R}^d \to \mathbb{R}$ and $\mu \in \mathcal{N}(\mathbb{R}^d)$, we write

$$\langle \mu, \varphi \rangle := \int \varphi \, d\mu.$$

Vague Topology

We endow $\mathcal{N}(\mathbb{R}^d)$ with the *vage* topology, in which a sequence $\{\mu_n\}$ converges to $\mu \in \mathcal{N}(\mathbb{R}^d)$ provided that

$$\langle \mu_n, \varphi \rangle \to \langle \mu, \varphi \rangle \text{ as } n \to \infty \text{ for any } \varphi \in C_c(\mathbb{R}^d).$$

Let us denote by $\mathfrak{N}$ the Borel σ-algebra corresponding to the vague topology.

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random point measure is a measurable mapping

$$\xi : (\Omega, \mathcal{F}) \to (\mathcal{N}(\mathbb{R}^d), \mathfrak{N}).$$

Notice that

- The probability measure $P(\cdot) = \mathbb{P}\xi^{-1}(\cdot)$ on $(\mathcal{N}, \mathfrak{N})$ is the **distribution** of ξ, and satisfies

$$P(M) = \left(\mathbb{P}\xi^{-1}\right)(M) = \mathbb{P}(\xi \in M) \quad \text{for all } M \in \mathfrak{N}.$$

- The **intensity** or expectation measure of ξ is the measure $\mathbb{E}\,\xi(\cdot)$ given by

$$(\mathbb{E}\,\xi)(M) = \mathbb{E}[\xi(M)] = \int_\Omega \xi(\omega, M)d\mathbb{P}, \quad \forall M \in \mathcal{B}(\mathbb{R}^d).$$

- The **Laplace functional** L_ξ of ξ is defined by

$$L_\xi(f) := \mathbb{E}\, e^{-\langle \xi, f\rangle}, \quad \forall f \in bM^+(\mathbb{R}^d).$$

(Same rôle as the Laplace transform for positive r.v.)

PROTOTYPES OF EQUATIONS

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) - Vu^2(t,x), \quad t > 0, \tag{1}$$

and

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \Delta u(t,x) + Vu^2(t,x), \quad t > 0, && (2) \\ u(0,x) &= \varphi(x) \geq 0, \quad x \in \mathbb{R}^d, && (3) \end{aligned}$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}, \quad V > 0.$$

In the first case,

$$\lim_{t \to \infty} \|u(t)\|_{L_1} \begin{cases} > 0 & \text{(persistence)} \\ = 0, & \text{(extinction)} \end{cases}$$

For the second equation,

$$\sup_{x \in \mathbb{R}^d} u(t,x) < \infty \ \forall t \quad \text{or} \quad \lim_{t \to T_0} u(t,x_0) = \infty, \ \text{some } T_0 < \infty, \ x_0 \in \mathbb{R}^d.$$

Critical dimension in both cases is $d = 2$:

For Equation (1), if $\varphi \not\equiv 0$, then $\lim_{t\to\infty} \|u(t)\|_{L_1} > 0$ iff $d > 2$,

For Equation (2), u blows up in finite time for any $\varphi \not\equiv 0$ if $d \leq 2$.

Systems of equations $\Longrightarrow$ Multitype populations

Other generators $\Longrightarrow$ Other particle motions

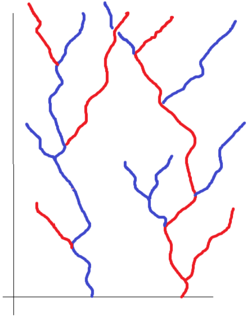
Other nonlinearities $\Longrightarrow$ Other branching laws

Persistence of a Multitype Branching System

(A) Particles in $\mathbb{R}^d$ of types $i \in \{1, \dots, k\}$

(B) Type i–particles live $\exp(V_i)$ lifetimes, $V_i > 0$, and move symmetric α_i–stable:

$$B_t^{(\alpha)} \overset{\mathcal{L}}{=} t^{1/\alpha} B_1^{(\alpha)}$$



(C) Branching numbers:

$$\mathbb{P}\mathrm{r}\left\{Z_{i1} = n_1, \dots, Z_{ik} = n_k\right\} = p^{(i)}(n_1, \dots, n_k)$$

Population space: counting measures on

$$\mathbb{S} \equiv \mathbb{R}^d \times \{1, \dots, k\};$$

as before, we write $\langle \mu, \phi \rangle = \int \phi(x)\mu(dx)$.

X_t : Random measure on $\mathbb{S}$ describing the population at time t.

$(X_t)_{0 \le t < \infty}$ is homogeneous Markov, with Laplace functional given by

$$\mathbb{E}\, e^{-\langle X_t, \phi \rangle} = e^{-\left\langle \Lambda_I, v_t^{\phi} \right\rangle}, \quad \phi \in C_c(\mathbb{S}, \mathbb{R}_+), \quad t \ge 0,$$

where v_t^{ϕ} is the mild solution of the nonlinear equation

$$\begin{aligned} \frac{\partial v_t(x,i)}{\partial t} &= \Delta_{\alpha_i} v_t(x,i) - V_i \Big[F_i \left(1 - v_t(x,1), \dots, 1 - v_t(x,k)\right) \\ &\qquad - \left(1 - v_t(x,i)\right) \Big], \qquad (4) \\ v_0(x,i) &= 1 - e^{-\phi(x,i)}, \quad \phi \in C_c(\mathbb{S}, \mathbb{R}_+). \end{aligned}$$

Here F_i is the *offspring generating function* of a type-i parent, i.e.

$$\begin{aligned} F_i(s_1, \ldots, s_k) &\equiv \mathbb{E}\left[s_1^{Z_{i1}} \cdots s_k^{Z_{ik}}\right] \\ &= \sum_{n_1, \ldots, n_k \geq 0} s_1^{n_1} \cdots s_k^{n_k} p^{(i)}(n_1, \ldots, n_k), \\ & \quad (s_1, \ldots, s_k) \in [0,1]^k. \end{aligned}$$

Assumptions

(A1) The mean matrix $M \equiv (m_{ij}) := (\mathbb{E}\, Z_{ij})$ is stochastic, with strictly positive entries. Hence M admits a unique equilibrum $l = (l_j)_{1\le j\le k}$; we put $\gamma_j := l_j/V_j$.

(A2) For all $i \in \{1,\dots,k\}$ the total offspring

$$Z_i := Z_{i1} + \cdots + Z_{ik}$$

is in the normal domain of attraction of a $1+\beta_i$–stable law, $0 < \beta_i \le 1$.
(The case $\beta_i = 1$ just means that Z_i has finite variance). Equivalently

$$F_i(s,\dots,s) \sim \text{Const.}(1-s)^{1+\beta_i} \text{ as } s \uparrow 1,\ i = 1,\dots,k.$$

(A3) The initial state X_0 is a Poisson population on $\mathbb{S}$ with intensity measure

$$\Lambda_l := \sum_{j=1}^{k} \gamma_j(\lambda \otimes \delta_j),$$

where λ denotes Lebesgue measure on $\mathbb{R}^d$.

Recall that, for any $t \geq 0$ and $\phi \in C_c(\mathbb{S}, \mathbb{R}_+)$,

$$\langle \mathbb{E}\, X_t, \phi \rangle = \langle \Lambda_I, \mathcal{U}_t \phi \rangle \,,$$

where $\mathcal{U} \equiv (\mathcal{U}_t)$ denotes the semigroup with generator

$$\mathcal{A}\phi(x,i) = \Delta_{\alpha_i}\phi(x,i) + V_i \sum_{j=1}^{k} (m_{ij} - \delta_{ij})\, \phi(x,j).$$

i.e., putting $w(t,x) := \mathcal{U}_t\phi(x)$, w solves the Cauchy problem

$$\begin{aligned} \frac{\partial w(t,x)}{\partial t} &= \mathcal{A}w(t,x), \quad t > 0, \\ w(0,x) &= \phi(x). \end{aligned}$$

Due to assumption **(A1)**, the measure Λ_I is invariant for the semigroup $\mathcal{U}$ and, therefore,

$$\langle \mathbb{E}\, X_t, \phi \rangle = \langle \Lambda_I, \mathcal{U}_t \phi \rangle = \langle \Lambda_I, \phi \rangle = \langle \mathbb{E}\, X_0, \phi \rangle \,.$$

Thus,

$$\mathbb{E}\, X_t = \mathbb{E}\, X_0, \quad t \geq 0.$$

It follows from a direct analysis of Equation (4) that

$$\begin{aligned} X_t &\overset{\mathcal{L}}{\longrightarrow} X_\infty \text{ as } t \to \infty, \\ \Lambda_\infty &:= \mathbb{E}\, X_\infty \leq \Lambda_I; \end{aligned}$$

moreover, either $\Lambda_\infty = 0$ or $\Lambda_\infty = \Lambda_I$. A criterion in this dichotomy is given by the following

THEOREM 1 The branching particle system started from X_0 is persistent in the sense that it converges as $t \to \infty$ towards a non-trivial equilibrium with the same intensity as that of X_0 if and only if $d > (\min \alpha_i)/(\min \beta_i)$.

Persistence of a class of nonlinear systems

We now use Theorem 1 to analyze the L^1–norm asymptotics of the mild solution of the nonlinear system

$$
\begin{aligned}
\frac{\partial v_i}{\partial t} &= \Delta_{\alpha_i} v_i - V_i \left[F_i\left(1 - v_1, \ldots, 1 - v_k\right) - \left(1 - v_i\right) \right], \\
v_i(x,0) &= f_i(x), \quad f_i \in C_c(\mathbb{R}^d, [0,1)), \quad i = 1, \ldots, k.
\end{aligned} \tag{5}
$$

This system possesses a unique global solution, which is positive for all t. The link between System (5) and the multitype model described before arises in the following way.
It is easy to see (e.g. using a renewal argument) that the solution of System (5) is given by

$$
v_i(x,t) = 1 - \mathbb{E} \exp\left\{ - \left\langle X_t^{(x,i)}, \Phi_f \right\rangle \right\}, \quad i = 1, \ldots, k, \quad t \geq 0,
$$

where

$$
\Phi_f(x,i) := -\log(1 - f_i(x)), \quad (x,i) \in \mathbb{S},
$$

and $(X_t^{(x,i)})$ stands for the system started from $\delta_{(x,i)}$.

Since $X_t \xrightarrow{\mathcal{L}} X_\infty$, it follows that, as $t \to \infty$,

$$\mathbb{E} \exp\{-\langle X_t, \Phi_f\rangle\} \to \mathbb{E} \exp\{-\langle X_\infty, \Phi_f\rangle\},$$

where, due to the fact that X_0 is Poisson distributed with intensity Λ_I,

$$\mathbb{E} \exp\{-\langle X_t, \Phi_f\rangle\} = e^{-\sum_{i=1}^k \gamma_i \langle \lambda, v_i(t)\rangle}.$$

Hence,

$$\lim_{t\to\infty} \sum_{i=1}^k \gamma_i \langle \lambda, v_i(t)\rangle = -\lim_{t\to\infty} \log \mathbb{E}\, e^{-\langle X_t, \Phi_f\rangle} = -\log \mathbb{E}\, e^{-\langle X_\infty, \Phi_f\rangle}, \tag{6}$$

which shows that the large time behavior of the numbers $\|v_i(t)\|_{L_1} = \langle \lambda, v_i(t)\rangle$, $i = 1, \ldots, k$, is determined by X_∞.
Indeed, (6) and the positivity of v_i yield

$$\lim_{t\to\infty} \|v_i(t)\|_{L^1} \le -((\mathrm{Const.}) \log \mathbb{E}\, e^{-\langle X_\infty, \Phi_f\rangle},$$

which, in case of $d \le (\min \alpha_i)/(\min \beta_i)$ and by Theorem 1, renders

$$\lim_{t\to\infty} \|v_i(t)\|_{L^1} = 0, \quad i = 1, \ldots, k.$$

The case $d > (\min \alpha_i)/(\min \beta_i)$

From the ergodicity of the type chain (which is implied by our assumptions on the F_i) it follows that

$$X_t^i \xrightarrow{\mathcal{L}} X_\infty \quad \text{as} \quad t \to \infty, \quad i = 1, \ldots, k,$$

where $(X_t^i)_{0 \le t < \infty}$ denotes the branching system starting from a Poisson population of type i-individuals X_0^i, with Lebesgue intensity.

Therefore, in the same way as in (6) it follows that

$$\lim_{t \to \infty} \|v_i(t)\|_{L^1} = -\log \mathbb{E}\, e^{-\langle X_\infty, \Phi_f \rangle},$$

which, again by Theorem 1, is strictly positive provided that at least on of $f_1, \ldots, f_k$ is not identically 0.

We summarize the above discussion into the following result.

THEOREM 2 Let $v_i(t) \equiv v_i(x,t)$, $i = 1, \ldots, k$, denote the solution components of the nonlinear system (5). Then, for each $i = 1, \ldots, k$,

$$\lim_{t\to\infty} \|v_i(t)\|_{L^1} = 0 \text{ if } d \leq \frac{\min \alpha_1}{\min \beta_i},$$

and

$$\lim_{t\to\infty} \|v_i(t)\|_{L^1} > 0 \text{ if } d > \frac{\min \alpha_1}{\min \beta_i},$$

provided that $f_i \not\equiv 0$ for at least one $i \in \{1, \ldots, k\}$.

Example Consider the nonlinear system

$$\begin{aligned}\frac{\partial v_1}{\partial t} &= \Delta_{\alpha_1} v_1 + V_1[(v_2 - v_1) - v_1 v_2]/2 \\ \frac{\partial v_2}{\partial t} &= \Delta_{\alpha_2} v_2 + V_2[(v_1 - v_2) - v_1 v_2]/2 \\ & v_i(0) = f_i \in C_c(\mathbb{R}^d, [0,1)), \quad i = 1, 2.\end{aligned}$$

Here $k = 2$,

$$F_1(s_1, s_2) = F_2(s_1, s_2) = \frac{1}{2} + \frac{s_1 s_2}{2},$$

and $\beta_1 = \beta_2 = 1$.
This system corresponds to a two-type critically branching population in which a type i-individual at rate V_i either dies without children with probability $1/2$, or with with complementary probability it has two children, one of each type.
Theorem 2 yields that, for any $(f_1, f_2) \not\equiv 0$ and $i = 1, 2$,

$$\lim_{t \to \infty} \|v_i(t)\|_{L^1} > 0 \text{ if and only if } d > \alpha_1 \wedge \alpha_2.$$

STABILITY OF SEMILINEAR

SYSTEMS OF EQUATIONS

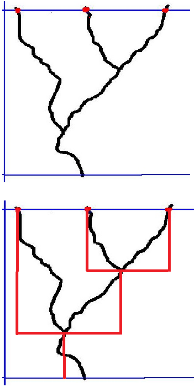
BRANCHING SYSTEM: Same as before, except that

(A) Two types of particles

(B) Type-i particles follow motions with generators A_i

(C) Branching numbers: $\delta_{(x,i)} \mapsto \beta_{i1}\delta_{(x,1)} + \beta_{i2}\delta_{(x,2)}$, where β_{ij} are fixed numbers,

(D) Starts with an individual $\delta_{(x,i)}$ at time $t = 0$.



Offspring tree

$\mathcal{T} \equiv (\mathcal{T}_t)_{t\geq 0}$: OFFSPRING TREE of the initial particle

$N_t^{i,\mathcal{T}} \equiv N_t^i := \#$ of type-i particles at time t

Define

$$S_t := V_1 \int_0^t N_s^1 \, ds + V_2 \int_0^t N_s^2 \, ds, \quad t \geq 0,$$

which is the weighted LENGTH OF $\mathcal{T}$ up to time t.

Consider the system

$$\begin{aligned} \frac{\partial u_t(x,1)}{\partial t} &= A_1 u_t(x,1) + V_1 u_t^{\beta_{11}}(x,1) u_t^{\beta_{12}}(x,2), \\ \frac{\partial u_t(x,2)}{\partial t} &= A_2 u_t(x,2) + V_2 u_t^{\beta_{21}}(x,1) u_t^{\beta_{22}}(x,2), \\ u_0(x,i) &= \varphi(x,i), \quad i = 1, 2, \ x \in \mathbb{R}^d, \end{aligned} \tag{7}$$

with $\varphi(\cdot, i) \in bB(\mathbb{R}^d, \mathbb{R}_+)$, $i = 1, 2$.

PROPOSITION For $x \in \mathbb{R}^d$ and $i = 1, 2$,

$$u_t(x,i) = \mathbb{E}\left[\mathrm{e}^{S_t} \prod_{(z,j) \in X_t^{(x,i)}} \varphi(z,j) \right], \ t \geq 0. \tag{8}$$

Existence of global solutions

(1) Expand our representation (8) in a series

$$u_t(x,i) = u_t^{(0)}(x,i) + u_t^{(1)}(x,i) + \cdots + u_t^{(k)}(x,i) + \cdots$$

in which, for $k = 0, 1, \ldots,$

$$u_t^{(k)}(x,i) = \mathbb{E}\left[\mathrm{e}^{S_t} \prod_{(z,j)\in X_t^{(x,i)}} \varphi(z,j); \sigma = k \right],$$

where $\sigma := \#$ of branchings occured in $[0,t)$.

(2) For any $\varphi \le 1$ and $k \ge 0$,

$$u_t^{(k)}(x,i) \le v_t^{(k)} T_t^i \varphi(x,i), \;\; t \ge 0, (x,i) \in \mathbb{S},$$

where $\left(T_t^i\right)_{t \ge 0}$ is the semigroup with generator A_i, $v_t^{(0)} \equiv 1$, and

$$v_t^{(k)} = \frac{\prod_{l=0}^{k-1} (1 + l(\mu^* - 1))}{k!} \left[V^* \int_0^t \left(\sup_{(z,i)} T_s^i \varphi(z,i) \right)^{\mu_* - 1} ds \right]^k,$$

for $k = 1, 2, \ldots$, with $V^* = V_1 \vee V_2$, and

$\mu^* = (\beta_{11} + \beta_{12}) \vee (\beta_{21} + \beta_{22}), \qquad \mu_* = (\beta_{11} + \beta_{12}) \wedge (\beta_{21} + \beta_{22})$.

(3) Thus,

$$u_t(x,i) \leq T_t^i \varphi(x,i) \left(1 + \sum_{k=1}^{\infty} v_t^{(k)} \right),$$

and it is easily shown that the series $\sum_{k=1}^{\infty} v_t^{(k)}$ is finite uniformly in t, provided that

$$V^* (\mu^* - 1) \int_0^{\infty} \left(\sup_{(z,i)} T_s^i \varphi(z,i) \right)^{\mu_* - 1} ds < 1. \tag{9}$$

Therefore:

COROLLARY For any $\varphi \in bB(\mathbb{S})_+$ bounded by 1 and satisfying condition (9) above, the corresponding mild solution of the non-linear system is global, and satisfies

$$u_t(x,i) \leq M T_t^i \varphi(x,i)$$

for some constant $M > 0$.

EXAMPLE 1 Consider the nonlinear equation

$$\begin{aligned} \frac{\partial u_t(x)}{\partial t} &= \Delta_\alpha u_t(x) + V u_t^\beta(x), \quad t > 0, \qquad (10) \\ u_0 &= \varphi \in bB(\mathbb{R}^d, \mathbb{R}_+), \ \beta \in \{2, 3, \ldots\}. \end{aligned}$$

Let $\gamma > 0$ and let $p_t^\alpha(x)$, $t > 0$ be the transition densities of the symmetric α-stable process in $\mathbb{R}^d$. If

(i) $d > \dfrac{\alpha}{\beta - 1}$, and

(ii) $\exists\ \delta > 0$ such that $0 \le \varphi(x) \le \delta p_\gamma^\alpha(x),\ x \in \mathbb{R}^d$, then the solution $u_t(x)$ of (10) is global, and

$$u_t(x) \le M p_{t+\gamma}^\alpha(x),\ x \in \mathbb{R}^d,\ t \ge 0$$

for some $M > 0$.

Indeed, by unimodality and scaling properties of stable densities,

$$\begin{aligned}
\sup_z T_s^\alpha \varphi(z) &= \sup_z \int_{\mathbb{R}^d} p_s^\alpha(z-y)\varphi(y)\,dz \\
&\leq \delta \int_{\mathbb{R}^d} p_s^\alpha(z-y) p_\gamma^\alpha(y)\,dy \\
&\leq \delta(\gamma+s)^{-d/\alpha} p_1^\alpha(0).
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^\infty \left(\sup_z T_s^\alpha \varphi(z)\right)^{\beta-1} ds &\leq C\delta \int_0^\infty (\gamma+s)^{-d(\beta-1)/\alpha}\,ds \\
&< \frac{1}{(\beta-1)V}
\end{aligned}$$

for $d > \alpha/(\beta-1)$ and sufficiently small $\delta > 0$, which yields Condition (9).

EXAMPLE 2 Consider the nonlinear system

$$\begin{aligned}
\frac{\partial u_t(x,1)}{\partial t} &= \Delta_{\alpha_1} u_t(x,1) + V_1 u_t^{\beta_{11}}(x,1) u_t^{\beta_{12}}(x,2), \\
\frac{\partial u_t(x,2)}{\partial t} &= \Delta_{\alpha_2} u_t(x,2) + V_2 u_t^{\beta_{21}}(x,1) u_t^{\beta_{22}}(x,2), \\
u_0(x,i) &= \varphi(x,i), \quad i = 1,\, 2,\ x \in \mathbb{R}^d, \quad \beta_{ij} \in \{1, 2, \ldots\}.
\end{aligned}$$

with $\varphi(\cdot, i) \in bB(\mathbb{R}^d, \mathbb{R}_+)$, $i = 1,\, 2$.

Proceeding as in the previous example, one can show the following: Assume

$$d > \frac{\max \alpha_i}{\min\{\beta_{i1} + \beta_{i2}\} - 1}.$$

Let γ_i, $i = 1,\, 2$, be given positive numbers.

There exist $\delta_i > 0$, $i = 1,\, 2$, such that if

$$0 \le \varphi(x,i) \le \delta_i p_{\gamma_i}^{\alpha_i}(x),\ x \in \mathbb{R}^d,\ i = 1,\, 2,$$

then the solution $(u_t(x,1), u_t(x,2))$ of the above nonlinear system is global. Moreover, there exists $M > 0$ satisfying

$$u_t(x,i) \le M p_{t+\gamma_i}^{\alpha_i}(x) \quad x \in \mathbb{R}^d, \quad t \ge 0,\ i = 1,\, 2.$$

Finite-time blow up of a single equation

For the equation

$$\begin{aligned}\frac{\partial u_t(x)}{\partial t} &= \Delta u_t(x) + u_t^2(x),\\ u_0(x) &= \varphi(x) \geq 0,\ x \in \mathbb{R}^d,\end{aligned}$$

it is known that, in dimensions $d = 1, 2$, u_t blows up in finite time for any nontrivial initial value φ. Let us consider the equation

$$\begin{aligned}\frac{\partial u_t(x)}{\partial t} &= Au_t(x) + Vu_t^\beta(x),\ \beta \in \{2, 3, \ldots\},\\ u_0 &= k1_B,\ k > 0,\end{aligned}$$

where $B \subset \mathbb{R}^d$ is a ball. The probabilistic representation yields

$$u_t(x) = \mathbb{E}\left[e^{S_t} \prod_{z \in X_t^x} 1_B(z)\right].$$

Therefore,

$$
\begin{aligned}
u_t(x) &= \mathbb{E}\left[\mathbb{E}\left(e^{S_t}\prod_{z\in X_t^x} 1_B(z)\,\middle|\,\mathcal{T}_t\right)\right] \\
&= \mathbb{E}\left[e^{S_t}\,\mathbb{E}\left(\prod_{z\in X_t^x} 1_B(z)\,\middle|\,\mathcal{T}_t\right)\right] \\
&= \mathbb{E}[e^{S_t}\underbrace{\mathbb{P}\mathrm{r}\left\{X_t^x(B) = N_t \,\middle|\, \mathcal{T}_t\right\}]}_{\text{lower estimates?}}
\end{aligned}
$$

LEMMA A Let $(T_t)_{t\geq 0}$ denote the semigroup generated by A. For any realization τ of $\mathcal{T}$ and any measurable, $f:\mathbb{R}^d\to\mathbb{R}_+$,

$$\mathbb{E}\left[\prod_{y\in X_t^x} f(y)\middle|\mathcal{T}_t=\tau_t\right]\geq \left(T_t f(x)\right)^{N_t^{\tau_t}},\ t\geq 0,\ x\in\mathbb{R}^d.$$

(Basically an application of Jensen's inequallity)

It follows that $u_t(x)\geq \mathbb{E}\left[e^{S_t}K^{N_t}\right]$ with $K\equiv K(t):=T_t 1_B(x)$. Let

$$h_t:=\mathbb{E}\left[e^{S_t}K^{N_t}\right].$$

Conditioning on the first branching time renders

$$h_t=K+V\int_0^t (h_s)^\beta\,ds,\quad t\geq 0,$$

and therefore, for $0 \leq t < V(\beta-1)^{-1}K^{1-\beta}$,

$$\mathbb{E}[e^{S_t}K^{N_t}] = \left(\frac{1}{K^{1-\beta} - Vt(\beta-1)}\right)^{1/(\beta-1)} .$$

LEMMA B For any $t > 0$ and $K > \dfrac{1}{Vt(\beta-1)}$,

$$h(t) = \mathbb{E}\left[e^{S_t}K^{N_t}\right] = \infty.$$

Corollary 1 (Nagasawa & Sirao). Let $u_t(x)$ solve the IVP

$$\begin{aligned}\frac{\partial u_t(x)}{\partial t} &= Au_t(x) + Vu_t^{\beta}(x),\ t>0,\\ u_0 &= f \in B(\mathbb{R}^d, \mathbb{R}_+),\end{aligned}$$

where $V > 0$ and $\beta \in \{2,3,\ldots\}$. If, for some x and $t > 0$, $T_t f(x) \geq \left(\frac{1}{Vt(\beta-1)}\right)^{1/(\beta-1)}$, then u blows up at x in finite time.

Example Let $w_t(x)$ solve the IVP

$$\begin{aligned}\frac{\partial w_t}{\partial t} &= \Delta_\alpha w_t + Vw_t^2,\ t>0,\quad 1<\alpha\leq 2,\\ w_0 &= \varphi \in B(\mathbb{R}, \mathbb{R}_+),\end{aligned}$$

where $\varphi \geq k1_B$, $k > 0$ and $B \subset \mathbb{R}$ open. Then for any $x \in B$, w_t blows up at x in finite time.

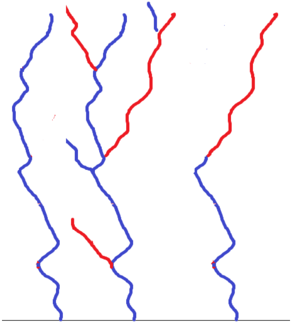
Indeed, let $x_0 \in B$ and B_0 a subinterval of B centered at x_0, and let W_t^{α,x_0} be the symmetric α-stable process at time $t \geq 0$, starting from $x_0 \in \mathbb{R}$. Then, for $f(x) = k1_{B_0}(x)$ and $t \geq 1$,

$$T_t f(x_0) = k\mathbb{P}\mathrm{r}\left\{W_t^{\alpha,x_0} \in B_0\right\} \geq \mathrm{Const}\, t^{-1/\alpha}$$

By Corollary 1, $w_t(x_0) = \infty$ for all $t \geq 1$ for which

$$\mathrm{Const}\, t^{(\alpha-1)/\alpha} \geq 1/V.$$

Blow up of a system of equations



$\partial\tau_t$: set of branches b_t of τ_t

$\left(W_s^{x,B_t}\right)_{0\leq s\leq t}$: process starting in x and following an A_i-motion along the edges of type i

$N_t := N_t^{1,\mathcal{T}} + N_t^{1,\mathcal{T}}, \;\; t \geq 0.$

Assumption on the motions:

T_t^i has a transition density $p_t^i(x-y)$, with symmetric unimodal $p_t^i(\cdot)$, $t>0$, $i=1,2$.

$$\tag{11}$$

LEMMA A' Under assumption (11), for any nonnegative, symmetric, unimodal, measurable f:

$$\mathbb{E}\left[\prod_{y\in X_t^{(x,i)}} f(y)\,\middle|\, \mathcal{T}_t=\tau_t\right]\geq \prod_{b_t\in\partial\tau_t}\mathbb{E}\left[f\left(W_t^{x,b_t}\right)\right].$$

$\mathbb{E}_{[n,m]}$: expectation when X_0 consists of n type-1 and m type-2 particles.

Recall our system:

$$
\begin{aligned}
\frac{\partial u_t(x,1)}{\partial t} &= A_1 u_t(x,1) + V_1 u_t^{\beta_{11}}(x,1) u_t^{\beta_{12}}(x,2), \\
\frac{\partial u_t(x,2)}{\partial t} &= A_2 u_t(x,2) + V_2 u_t^{\beta_{21}}(x,1) u_t^{\beta_{22}}(x,2), \\
u_0(x,i) &= \varphi(x,i), \quad i = 1,\, 2,\; x \in \mathbb{R}^d,
\end{aligned} \tag{12}
$$

LEMMA B' Let $2 \le \beta_{11} + \beta_{12} \le \beta_{21} + \beta_{22}$.

(1) If $\beta_{11} + \beta_{12} = \beta_{21} + \beta_{22}$ or $\beta_{11} \ge 2$, then,

$$\mathbb{E}_{[1,0]}\left[\mathrm{e}^{S_t} K^{N_t} \right] = \infty \text{ for } K \ge c\, t^{-1/(\beta_{11}+\beta_{12}-1)},\; t > 0.$$

(2) If $\beta_{11} = \beta_{22} = 0$, then,

$$\mathbb{E}_{[1,0]}\left[\mathrm{e}^{S_t} K^{N_t} \right] = \infty \text{ for } K \ge c' t^{-2/(\beta_{12}+\beta_{21}-2)},\; t > 0.$$

Here c and c' are independet of t.

1. Assume $\varphi(\cdot,1)=\varphi(\cdot,2)=f(\cdot)$, with f as in Lemma A'.

Let $\left(Z_t^1\right)_{t\geq 0}$ and $\left(Z_t^2\right)_{t\geq 0}$ be Markov processes in $\mathbb{R}^d$ (starting at 0), with generators A_1 and A_2, respectively.

Take $K := \inf_{0\leq r\leq t} \mathbb{E}\left[f\left(x+Z_r^1+Z_{t-r}^2\right)\right]$. Then,

$$
\begin{aligned}
u_t(x,1) &= \mathbb{E}_{[1,0]}\left[\mathrm{e}^{S_t}\prod_{y\in X_t^{(x,1)}} f(y)\right] \\
&= \mathbb{E}_{[1,0]}\left[\mathrm{e}^{S_t}\,\mathbb{E}_{[1,0]}\left(\prod_{y\in X_t^{(x,1)}} f(y)\,\middle|\,\mathcal{T}\right)\right] \\
\text{Lemma A'} \quad &\geq \mathbb{E}_{[1,0]}\left[\mathrm{e}^{S_t}K^{N_t}\right] \\
&= \infty \text{ if } K \text{ meets Lemma B' (1) or (2).}
\end{aligned}
$$

2. If $\varphi(\cdot,1) \neq \varphi(\cdot,2)$, assume, in addition to assumption (11), that

For all $t > 0$, $p_t^i(\cdot)$ is strictly positive and continuous, $i = 1, 2$.

Let $\varphi(\cdot,i) \geq k_i 1_{B_i}(\cdot)$ for some constants $k_i > 0$ and balls B_i, $i = 1, 2$.

Then, because of the assumed positivity and continuity of p_t^i, for any fixed $t_0 > 0$,

$$u_{t_0}(\cdot,1) \wedge u_{t_0}(\cdot,2) \geq k 1_B \equiv f,$$

for some $k > 0$, where B is th e unit ball in $\mathbb{R}^d$.

Restarting the nonlinear system at time $t = t_0$ if necessary, one can assume that

$$\varphi(\cdot, 1) \wedge \varphi(\cdot, 2) \geq f$$

and proceed as in step **1**.

In this way we obtain the following

THEOREM Suppose that for each ball $B \subset \mathbb{R}^d$ centered at the origin,

$$K := \inf_{0 \leq r \leq t} \mathbb{P}\mathrm{r}\{Z_r^1 + Z_{t-r}^2 \in B\}$$

meets the conditions of Lemma B' (1) or (2). Under the above assumptions on $p_t^i(\cdot)$, $t \geq 0$, $i = 1$, 2, the solution to system (12) exhibits blow up in finite time for all initial values $\varphi(x, i)$ satisfying

$$\varphi(x, i) \geq k_i 1_{B_i}, \; x \in \mathbb{R}^d,$$

for some constants $k_i > 0$ and balls $B_i \subset \mathbb{R}^d$, $i = 1, 2$.

Example 1. Consider the system

$$\begin{aligned}\frac{\partial u_t}{\partial t} &= \Delta_{\alpha_1} u_t + V_1 u_t v_t\\ \frac{\partial v_t}{\partial t} &= \Delta_{\alpha_2} v_t + V_2 u_t v_t\\ u_0(x) &= \varphi_1(x),\ v_0(x) = \varphi_2(x),\ x \in \mathbb{R}^d,\end{aligned}$$

where $\varphi_i \in B(\mathbb{R}^d, \mathbb{R}_+)$ and $V_i > 0$, $i = 1, 2$.

Let us denote by $(S_t^\alpha)_{t\geq 0}$ the symmetric α-stable process in $\mathbb{R}^d$ with $S_0^\alpha = 0$.

Then, for any ball $B \subset \mathbb{R}^d$ centered at the origin, there exists $c_0 > 0$ such that

$$\mathbb{P}\mathrm{r}\left\{S_r^{\alpha_1} + S_{t-r}^{\alpha_2} \in B\right\} \geq c_0 t^{-d/\min\{\alpha_1,\alpha_2\}} \quad \text{for all } r \in [0, t].$$

Here $K := c_0 t^{-d/\min\{\alpha_1,\alpha_2\}}$ meets conditions of Lemma B' (1) iff $d < \min\{\alpha_1, \alpha_2\}$. Therefore,

For $d = 1$ and $\min\{\alpha_1, \alpha_2\} > 1$ the system blows up in finite time for any (φ_1, φ_2) for which $\varphi_i \geq k_i 1_{B_i}$.

Example 2. Consider the system

$$\begin{aligned}\frac{\partial u_t}{\partial t} &= \Delta_{\alpha_1} u_t + v_t^p \\ \frac{\partial v_t}{\partial t} &= \Delta_{\alpha_2} v_t + u_t^q \\ u_0(x) &= \varphi_1(x),\ v_0(x) = \varphi_2(x),\ x \in \mathbb{R}^d,\end{aligned}$$

where $\varphi_i \in B(\mathbb{R}^d, \mathbb{R}_+)$ and $(p,q) \in \{(2,2),(2,3)\}$.
Now $K := c_0 t^{-d/\min\{\alpha_1,\alpha_2\}}$ meets conditions of Lemma B' (2) iff

$$\frac{1}{2}(p+q-2) < \frac{\min\{\alpha_1,\alpha_2\}}{d}.$$

Hence, any nontrivial solution of the above system blows up in finite time, provided that

$$p = q = 2,\ d = 1 \text{ and } \min\{\alpha_1,\alpha_2\} > 1,$$

or

$$p = 2,\ q = 3,\ d = 1 \text{ and } \min\{\alpha_1,\alpha_2\} > \frac{3}{2}.$$

References

Fujita, H. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I* Vol. 13 (1966), 109-124.

Ikeda, N., Nagasawa, M., Watanabe, S. Branching Markov Processes I, II, III. *J. Math. Kyoto Univ.* Vol. 8 (1968), 233-278 and Vol. 9 (1969), 95-160.

Levine, H., The role of critical exponents in blow-up theorems. *SIAM Rev.* Vol. 32 (1990), 262-288.

López-Mimbela, J.A. A probabilistic approach to existence of global solutions of a system of nonlinear differential equations. *Aportaciones Mat. Notas Investigación* Vol. 4 (1996), 147-155.

López-Mimbela, J.A. and Torres-Rodríguez, A. Intrinsic ultracontractivity and blowup of a smilinear Dirichlet boundary value problem. *Aportaciones Mat. Notas Investigación* Vol. 14 (1998), 283-290.

López-Mimbela, J.A. and Wakolbinger A. Lenght of Galton-Watson trees and blow-up of semilinear systems. *J. Appl. Prob.* Vol. 35 (1998), 802-811.

López-Mimbela, J.A. and Wakolbinger, A. A probabilistic proof of non-explosion of a non-linear PDE system. *J. Appl. Probab.* Vol. 37 (2000), 635–641.

Nagasawa, M., Sirao, T. Probabilistic treatment of the blowing up of solutions for a nonlinear integral equation. *Trans. Amer. Math. Soc.* Vol. 139 (1969), 301-310.

Samarskii, A., Galaktionov, V., Kurdyumov, S., Mikhailov, A. *Blow-up in Quasilinear Parabolic Equations.* Trans. by Michael Grinfeld. De Gruyter Expositions in Mathematics 19. De Gruyter, Berlin 1995.