# *Statistical methods for financial models driven by Lévy processes*

**José Enrique Figueroa-López**

**Department of Statistics, Purdue University**

**PASI**

**Centro de Investigación en Matemátics (CIMAT)**

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## Objectives

To *introduce* current advances and problems in the theory of

1. *Financial models with jumps* and

2. *related statistical inference*.

## Program

**I.** Background on Lévy processes

**II.** Introduction to financial models driven by Lévy processes

**III.** Classical statistical methods

**IV.** Recent nonparametric methods based on low- and high-frequency sampling

# Part I: Background on Lévy processes

# Lévy processes

A collection of real-valued random variables $X_t(\omega) : \Omega \to \mathbb{R}$, indexed by time $t \geq 0$ such that

1. $X_0 = 0$;
2. *(Independent Increments)*:
   $X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ independent for any $t_0 < \cdots < t_n$;
3. *(Stationary Increments)*:
   $X_t - X_s \stackrel{\mathfrak{D}}{=} X_{t-s}$ for any $0 \leq s \leq t$;
4. *(Cádlág paths)*: For every state $\omega \in \Omega$, the path $t \to X_t(\omega)$ is s.t.
   $X_{t-} := \lim_{s \nearrow t} X_s$ exists and $X_{t+} := \lim_{s \searrow t} X_s = X_t$;
5. *(No fixed jumps)*:
   $\mathbb{P}(\Delta X_t \neq 0) = 0$, for any $t > 0$, where $\Delta X_t := X_t - X_{t-}$;

# Fundamental examples

**Deterministic (non-random):**

$X_t = tb$ for any $t$ where $b$ is constant (the drift);

**Wiener process or Brownian Motion:** $\{W_t\}$

Independent and stationary increments with $\underline{W_t \sim \mathcal{N}(0,t)}$ and <u>continuous paths</u>;

Relevance: Model of choice to describe a random quantity that is the result of many shocks of small magnitude (*Invariance Theorem*).

**Poisson process:** $\{N_t\}$

Cádlág, Independent, and Stationary Increments with $N_t \sim$ Poisson$(\lambda t)$;

Relevance: Model of choice to count events occurring independently of one another and homogeneously through time with an intensity of $\lambda$ events per unit time;

# Fundamental examples. Cont...

## Compound Poisson Process:

- Let $J_1, \ldots, J_n$ be independent copies of a random variable $J$;
- For each $A \subset \mathbb{R}$, let $\rho(A) = \mathbb{P}(J \in A)$ (the distribution of $J$);
- $\{N_t\}$ be a Poisson process with intensity $\lambda$, independent of $\{J_i\}$;

$\{X_t\}$ is compound Poisson if for any $t \geq 0$

$$X_t := \sum_{i=1}^{N_t} J_i;$$

Interpretation: Each $J_i$ represents a "shock" and $N_t$ determines the arrival times of the shocks; Inter-arrival times are i.i.d. exponential with mean $\lambda$;

Terminology: $\nu(A) := \lambda\rho(A)$ is called the Lévy measure of the process;

## Characterization of the marginal distributions

**Lévy-Khintchine Representation:** The characteristic function (c.f.) of $X_t$ is given by

$$\mathbb{E}e^{iuX_t} = e^{t\left\{iub - \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}_0} (e^{iux} - 1 - iux 1_{|x|\leq 1})\nu(dx)\right\}},$$

for some constants $\sigma \geq 0$ and $b \in \mathbb{R}$, and a measure $\nu$ on $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$ such that

$$\int_{\mathbb{R}_0} (|x|^2 \wedge 1)\nu(dx) < \infty.$$

Furthermore, the triple $(b, \sigma^2, \nu)$ is uniquely determined, and any such triplet is possible.

Notation: $\nu$ is called the Lévy measure of the process.

## Back to the fundamental examples

**Deterministic Lévy**: $X_t = bt$ has c.f. $\mathbb{E}e^{iuX_t} = e^{ibt}$.

**Wiener process**: $X_t = W_t$ has c.f. $\mathbb{E}e^{iuX_t} = e^{-\frac{u^2 t^2}{2} t}$.

**Compound Poisson**: $X_t = \sum_{i=1}^{N_t} J_i$ has c.f. $\mathbb{E}e^{iuX_t}$ is given by

$$\sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \mathbb{E} e^{iu \sum_{i=1}^{n} J_i} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\left(\lambda t \int e^{iux} \rho(dx)\right)^n}{n!}$$

$$= \exp\left\{ t \int \left(e^{iux} - 1\right) \lambda \rho(dx) \right\},$$

Lévy Triplet: $\sigma = 0$, $\nu(dx) = \lambda \rho(dx)$, $b = \lambda \int_{|x| \le 1} x \rho(dx)$.

Claim: For any measure $\nu$ such that $\nu\big(\mathbb{R}_0\big) < \infty$, there exists a compound Poisson process $X$ with Lévy measure $\nu$;

# Construction of a General Lévy Process

Problem: Suppose we want to simulate a Lévy process with triplet $(b, \sigma, \nu)$.

General approximation algorithm:

1. *(Brownian Component)*.

$$\sigma W_t \approx \sigma n^{-1/2} \sum_{i=1}^{[nt]} \varepsilon_i, \quad \varepsilon_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1).$$

2. *(Big-Jumps Component)*. Compound Poisson with Lévy measure $\nu^{cp}(dx) := \mathbf{1}_{|x|\geq 1}\nu(dx)$:

$$X_t^{cp} = \sum_{i=1}^{N_t} J_i, \qquad \begin{array}{l} \{N_t\} \sim \text{Poisson with } \lambda := \nu(|x| \geq 1), \\ J_i \overset{\text{i.i.d.}}{\sim} \rho^{cp}(dx) := \mathbf{1}_{|x|\geq 1}\nu(dx)/\lambda. \end{array}$$

3. *(Small-jump Component).* Compensated compound Poisson with Lévy measure $\nu^\varepsilon(dx) := \mathbf{1}_{\varepsilon \le |x| < 1}\nu(dx) \quad (\varepsilon > 0)$:

$$\bar{X}_t^\varepsilon := \sum_{i=1}^{N_t^\varepsilon} J_i^\varepsilon - \mathbb{E}\sum_{i=1}^{N_t^\varepsilon} J_i^\varepsilon,$$

where $\begin{cases} \{N_t^\varepsilon\} \sim \text{Poisson with } \lambda_\varepsilon := \nu(\varepsilon \le |x| < 1), \\ J_i^\varepsilon \overset{\text{i.i.d.}}{\sim} \rho^\varepsilon(dx) := \mathbf{1}_{\varepsilon \le |x| < 1}\nu(dx)/\lambda_\varepsilon. \end{cases}$

**Theorem** *The process* $X_t^\varepsilon := bt + \sigma W_t + X_t^{cp} + \bar{X}_t^\varepsilon$ *converges (in distribution) as* $\varepsilon \searrow 0$ *to a Lévy process* $\{X_t\}_{t\ge 0}$ *with Lévy triplet* $(b, \sigma, \nu)$*. Furthermore,*

$$|\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(X_T^\varepsilon)| \le \|\varphi'\|_\infty \sqrt{T\int_{|x|\le\varepsilon} x^2\nu(dx)}.$$

# Moments

1. If $g : \mathbb{R} \to \mathbb{R}_+$ is locally bounded submultiplicative or subadditive (i.e. $g(x+y) \le Kg(x)g(y)$ or $g(x+y) \le K(g(x)+g(y))$, $\forall x, y$), then
$$\mathbb{E}g(X_t) < \infty \quad \Longleftrightarrow \quad \int_{|x|\ge 1} g(x)\nu(dx) < \infty.$$
2. $\mathbb{E}|X_t|^k < \infty \quad \Longleftrightarrow \quad \int_{|x|\ge 1} |x|^k \nu(dx) < \infty, \quad (k \ge 1)$.
3. By formal differentiating of the c.f., we have the following
   (a) $\mathbb{E}X_t = t\mathbb{E}X_1 = t\left(b + \int_{|x|\ge 1} x\nu(dx)\right)$;
   (b) $\mathrm{Var}(X_t) = t\mathrm{Var}(X_1) = t\left(\sigma^2 + \int x^2\nu(dx)\right)$;
   (c) $\mathrm{Skew}(X_t) = \frac{\mathbb{E}(X_t - \mathbb{E}X_t)^3}{\mathrm{Var}(X_t)^{3/2}} = \frac{\mathrm{Skew}(X_1)}{\sqrt{t}}$;
   (d) $\mathrm{Kurt}(X_t) = \frac{\mathbb{E}(X_t - \mathbb{E}X_t)^4}{\mathrm{Var}(X_t)^2} - 3 = \frac{\mathrm{Kurt}(X_1)}{t}$;

# Small-time moment asymptotics

1. [FL08]: Let $\varphi : \mathbb{R} \to \mathbb{R}$ be locally bounded $\nu$-continuous such that

   (a) $|\varphi| \leq g$ for a subm. or subad. $g$ s.t. $\int_{|x|\geq 1} g(x)\nu(dx) < \infty$;

   (b) $\lim_{x\to 0} x^{-2}\varphi(x) = 0$;

   Then,

$$\lim_{t\searrow 0} \frac{1}{t}\mathbb{E}\varphi(X_t) = \int \varphi(x)\nu(dx) := \nu(\varphi).$$

2. In particular, we recover the well-known limit

   $\lim_{t\searrow 0} \frac{1}{t}\mathbb{P}\left(X_t \geq a\right) = \nu([a,\infty)),$ for $a > 0$ s.t. $\nu(\{a\}) = 0$.

3. [FLH09]: If $\nu(dx) = s(x)dx$ and $s$ is $C^2$ in a neighborhood of $a$, then

$$\lim_{t\searrow 0} \frac{1}{t}\left(\frac{1}{t}\mathbb{P}\left(X_t \geq a\right) - \nu([a,\infty)\right) := d_2(a) \in (0,\infty).$$

# Lévy-Itô path decomposition

Let $X : \Omega \to \mathbb{R}$ be a Lévy process with triplet $(b, \sigma^2, \nu)$. There exist mutually independent

- Wiener process $W$;
- Compound Poisson process $X^{cp}$ with Lévy measure $\nu^{cp}(dx) = \mathbf{1}_{|x|\geq 1}\nu(dx)$;
- Compound Poisson process $X^{\varepsilon}$ with Lévy measure $\nu^{\varepsilon}(dx) = \mathbf{1}_{\varepsilon<|x|<1}\nu(dx)$;

(a.s.) <u>defined on $\Omega$</u> such that, for almost every $\omega$ (a.k.a. “path of $X$”),

$$X_t(\omega) = bt + \sigma W_t + X_t^{cp} + \lim_{\varepsilon \searrow 0} \left( X_t^{\varepsilon} - \mathbb{E}X_t^{\varepsilon} \right), \quad \forall t \geq 0.$$

# Some Consequences

1. For any closed interval $A = [a, b]$ in $\mathbb{R}_0$,

$$N_t(A) := \sum_{s \leq t} \mathbf{1}_{\{\Delta X_s \in A\}} = \# \, \{\text{Jumps of size in } A \text{ before } t\},$$

   is a Poisson process with intensity $\nu(A)$; Hence,

$$\nu(A) = \text{ Expected number of jumps of size in } A \text{ per unit time.}$$

2. For any two mutually disjoint closed intervals $A_1, A_2 \subset \mathbb{R}_0$,

$$N_{\cdot}(A_1) \text{ and } N_{\cdot}(A_2) \text{ are independent Poisson processes;}$$

## Some Consequences. Cont...

1. The paths of $X$ are continuous if and only if $X = bt + \sigma W_t$ (Brownian Motion with drift);

2. The paths of $X$ are non-decreasing (subordinator) if and only if

$$\sigma = 0, \quad \nu((-\infty, 0)) = 0, \quad b - \int_0^1 x\nu(dx) \geq 0.$$

3. The paths of $X$ are of bounded variation (namely, the difference of two increasing processes) if and only if

$$\sigma = 0, \quad \text{and} \quad \int_{|x|\leq 1} |x|\nu(dx) < \infty.$$

# Accurate simulation of Lévy processes

**Key Idea:** (Asmussen & Rosiński 2001) Approximate the small jumps of a Lévy process by a Wiener process;

**General Procedure:**

1. *(Big-Jumps Component)*.
   $X_t^{cp,\varepsilon}$ = Compound Poisson with Lévy measure
   $\nu^{cp,\varepsilon}(dx) := \mathbf{1}_{|x|\geq\varepsilon}\nu(dx)$.

2. *(Approximation of small-jumps by a Brownian motion $\{B_t\}$)*.

$$\mathcal{R}_t^{\varepsilon} := X_t - (\sigma W_t + X^{cp,\varepsilon} - tb_\varepsilon) \approx \sigma_\varepsilon B_t$$

where

$$b_\varepsilon := \int_{\varepsilon\leq|x|<1} x\nu(dx), \quad \sigma_\varepsilon^2 := \int_{|x|<\varepsilon} x^2\nu(dx).$$

**Theorem (Asmussen & Rosiński 2001, Cont & Tankov, 2004)**

*The process*

$$\hat{X}_t^\varepsilon := \sigma W_t + X_t^{cp,\varepsilon} + t(b - b_\varepsilon) + \sigma_\varepsilon B_t$$

*converges (in distribution) as $\varepsilon \searrow 0$ to a Lévy process $\{X_t\}_{t\geq 0}$ with Lévy triplet $(b, \sigma, \nu)$ if the following condition holds:*

$$\frac{\sigma_\varepsilon}{\varepsilon} \to \infty, \qquad (\varepsilon \to 0).$$

*Furthermore, defining $\rho_\varepsilon := \int_{|x|\leq\varepsilon} |x|^3 \nu(dx)/\sigma_\varepsilon^3$,*

$$|\mathbb{E}\varphi(X_T) - \mathbb{E}\varphi(\hat{X}_T^\varepsilon)| \leq A\|\varphi'\|_\infty \rho_\varepsilon \sigma_\varepsilon \leq A\|\varphi'\|\varepsilon,$$

*for an absolute constant $A$ ($\leq 20$).*

# Summary of Properties

## Parameters and characterization

- The statistical law of the whole process is determined by the marginal distribution of $X_1$
- Three parameters, two reals $\sigma^2$, $b$, and a measure $\nu(dx)$, so that

$$
\begin{aligned}
&(1)\quad Ee^{iuX_1} = \exp\{b - \tfrac{\sigma^2 u^2}{2} + \textstyle\int (e^{iux} - 1 - iux\mathbf{1}_{|x|\le 1})\nu(dx)\} \\
&(2)\quad X_t = bt + \sigma W_t + \underbrace{X_t^{cp}}_{\text{Jmp Size}\ge 1} + \lim_{\varepsilon\downarrow 0} \underbrace{\{X_t^{\varepsilon} - tb_{\varepsilon}\}}_{\text{Jmp Size}\in[\varepsilon,1)}, \\
&(3)\quad N_t([a,b]) := \textstyle\sum_{u\le t} \mathbf{1}[\Delta X_u \in [a,b]] \sim \text{Poisson}\,(t\nu([a,b])) \\
&(4)\quad \lim_{t\searrow 0} \tfrac{1}{t}\mathbb{E}\varphi(X_t) = \nu(\varphi) := \textstyle\int \varphi(x)\nu(dx);
\end{aligned}
$$