Stat 331/Elec 331, Final Exam, due December 19, 5 pm.

Solutions should be clear, complete and easy to follow. You are allowed to use the book, lecture notes but no previous homework or solutions. Collaboration is not allowed. The time limit is five hours. Each problem is worth 6 points. Late turn-ins are not accepted.

1. The continuous random variable $X$ has pdf

$$f(x) = \begin{cases} 
\frac{1}{2} & \text{if } 0 < x \leq 1 \\
\frac{1}{2x^2} & \text{if } x \geq 1 
\end{cases}$$

a. Show that this is a possible pdf for a continuous random variable.

b. Compute $E[X]$.

c. Find the cdf of $X$ and sketch its graph.

d. Let $Y = 1/X$. Find the cdf of $Y$ and sketch its graph.

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2. A company manufactures metal plates of size 5 x 10 (inches). Due to random fluctuations, a manufactured plate has a size of $X \times Y$ inches where $(X, Y)$ follows a bivariate normal distribution with means 5 and 10, variances 0.01 and 0.04 and correlation coefficient 0.8.

a. A plate is useless if its circumference is less than 29 or more than 31 inches. Suppose a plate has width $X = 5.5$. Given this, what is the probability that it is useless?

b. What is the expected area of a plate? You do not need to compute any integrals to solve this.

c. Suppose that you have three plates with circumferences $C_1$, $C_2$ and $C_3$ respectively. What is the probability that $C_1 + C_2 > 2C_3$?
3. The random variable $X$ has pdf

$$f(x) = (a + 1)x^a, \quad 0 \leq x \leq 1$$

where $a$ is an unknown parameter.

a. Find the maximum likelihood estimator and the method of moments estimator of $a$ based on a sample $X_1, X_2, ..., X_n$.

b. Suppose $a = 2$. Describe how you can simulate observations on $X$ based on observations from a uniform $[0,1]$-distribution. If such a uniform value is 0.008, what value of $X$ does this give?

4. Accidents on a certain road occur according to a Poisson process with rate $\lambda$ accidents/week.

a. The two towing companies $A$ and $B$ have agreed to take turns in dealing with the accidents. Thus, $A$ takes care of the first accident, $B$ the second and so on. Consider the process of accidents that $A$ takes care of. Is this a Poisson process? If so, what is the rate?

b. Suppose that in a particular year, it is observed that $N$ of the 52 weeks had no accidents. What is the distribution of $N$ (name and parameters)?

c. Based on $N$, suggest a reasonable estimator of $\lambda$. You do not have to use any particular method if you don’t want to.

5. Customer groups arrive to a service station according to a Poisson process with rate $\alpha$ groups/minute. With probability $p$, such a group consists of a single individual and with probability $1 - p$ it consists of a pair. There is a
single server and room for two to wait in line (so the state space is \{0, 1, 2, 3\}). If a pair arrives and they cannot both join, they both leave.

a. Describe the system in a rate diagram.

b. Suppose \( \alpha = \beta \) and \( p = 1/2 \). State the balance equations and find the stationary distribution \((\pi_0, \pi_1, \pi_2, \pi_3)\).

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6. Consider the plots on the next page. Each is a plot of 1000 simulated observations on a pair of random variables \((X,Y)\). Which of them do you think come from bivariate normal distributions? For those that do, what can you say about the correlation coefficient \( \rho \) (negative, positive, zero, small, large etc)?

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7. As you may know, there is an element named after California, element number 98, californium. It’s perhaps more interesting that as many as four elements are named after a small village in a northern European country. Which country, village and elements? No points for this questions, just the pleasure of acquiring obscure facts. Yes, you may use the Internet and no, this is not included in the time limits.