Final exam, Stat/Elec 331, Fall 2002

Solutions should be clear, complete and easy to follow. You are allowed to use the book, a calculator, your lecture notes and lecture notes posted on the course web page. The time limit is five hours. Each problem is worth 6 points. The deadline to turn in the exam is December 18, 5 pm. Late turn-ins are not accepted.

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1. The random variable $X$ has pdf $f(x) = e^{-x}, x > 0$. Let $Y = e^X$ and $Z = 1/X$. Find the pdf’s of $Y$ and $Z$ and sketch their graphs.

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2. A company manufactures metal plates of size 5 x 10 (inches). Due to random fluctuations, a manufactured plate has a size of $X \times Y$ inches where $(X, Y)$ follows a bivariate normal distribution with means 5 and 10, variances 0.01 and 0.04 and correlation coefficient 0.8. Let $C$ be the circumference (perimeter) and $A$ the area of a plate

   a. Find $E[C]$ and $E[C|X = x]$.

   b. Find $E[A]$ and $E[A|X = x]$.

   c. A plate is useful if $29 \leq C \leq 31$. What is the probability that a plate is useful?

   d. Now suppose that you have ten plates. What is the probability that at least nine are useful?

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3. The random variable $X$ has pdf
\[ f(x) = axe^{-ax^2}, \quad x \geq 0 \]

where \( a \) is an unknown parameter.

a. Find the maximum likelihood estimator of \( a \) based on a sample \( X_1, X_2, ..., X_n \).

b. Suppose \( a = 2 \) and that you have a random variable \( U \) which is uniform on \((0, 1)\). If you want to simulate an observation on \( X \), which function of \( U \) should you use? If \( U = 0.5 \), what value do you get for \( X \)?

4. Customer groups arrive to a service station according to a Poisson process with rate \( \alpha \) groups/minute. With probability \( p \), such a group consists of a pair and with probability \( 1-p \) it consists of a single individual. There is a single server and room for two to wait in line (so the state space is \( \{0, 1, 2, 3\} \)). Service times are exponential with rate \( \beta \) (mean \( 1/\beta \)). If a pair arrives and they cannot both join, with probability \( q \) they both leave and with probability \( 1-q \), one stays and the other leaves.

a. Describe the system in a rate diagram.

b. Let \((\pi_0, ..., \pi_3)\) denote the stationary distribution and assume that the system is in equilibrium. Express the expected proportion of lost customers as a function of \( \pi_0, ..., \pi_3, p \) and \( q \).

c. Suppose \( \alpha = \beta \) and \( p = q = 1/2 \). State the balance equations and find the stationary distribution \((\pi_0, \pi_1, \pi_2, \pi_3)\).

5. Consider the plots on the last page. Each is a plot of 1000 simulated observations on a pair of random variables \((X,Y)\). Which of them do you think come from bivariate normal distributions? For those that don’t, argue
why. For those that do, what can you say about the correlation coefficient $\rho$ (negative, positive, zero, small, large etc)? For some reason, some plots have diagonal lines in them; just disregard those.

6. A company manufactures metal rods which are supposed to be 30 inches long. They are produced in a two-stage process: First the rod is roughly cut in a machine, then it is measured and finished off by hand to be exactly 30 inches long. Due to random fluctuations, the machine produces rods whose lengths are normally distributed with mean $\mu$ inches and standard deviation $\sigma = 1$. The mean $\mu$ can be chosen in advance and the problem is to choose an optimal $\mu$. First note that, if a rod is shorter than 30, it is useless and wasted. Therefore, $\mu$ must be set to be at least 30, to lower this risk. However, if $\mu$ is set too high, too much material will be wasted in the finishing. Let $W$ be the amount of wasted material and find the value of $\mu$ that minimizes $E[W]$, the expected waste.

Hints: If a length is $L$, then $W$ is either $L$ or $L - 30$, depending on what $L$ is. First show that $E[W] = \mu + 30F_L(30) - 30$ where $F_L$ is the cdf of $L$. To minimize in $\mu$, recall that $\Phi'(x) = \varphi(x)$ (cdf and pdf of the standard normal distribution).

7. Bonus question, not included in the time limits. As you may know, there is an element named after California, element number 98, californium. It’s perhaps more interesting that as many as four elements are named after a small village in a northern European country. Which country, village and elements? No points for this questions, just the pleasure of acquiring obscure facts. Google allowed.