Stat 331/Elec 331, Solutions to Homework 1

1a. $E \cap F^c$, $P(E \cap F^c) = P(E) - P(E \cap F) = 0.5 - p$.

b. $(E \cap F^c) \cup (E^c \cap F)$, has probability $(0.5 - p) + (0.5 - p) = 1 - 2p$ by part **a** and symmetry.

c. $(A \cup B)^c$ (which can also be written as $A^c \cap B^c$). This has probability $1 - P(A \cup B) = 1 - (0.5 + 0.5 - p) = p$.

2. $\binom{6}{2}/\binom{10}{3} = 1/8$. The number of ways to choose three numbers among ten is $\binom{10}{3}$. The number of ways to choose three numbers so that the smallest of them is 4 is $\binom{6}{2}$. This is so because 4 has to be chosen and then there are $\binom{6}{2}$ ways o choose the remaining two numbers among the numbers 5,...,10.

3. Let A be the event of interest and condition on $B_1 = \{$ the three A's are chosen$\}$, $B_2 = \{$ two A's and some other letter are chosen$\}$, $B_3 = \{$ two S's and some other letter is chosen$\}$ and $B_4 = \{$ all three letters are different$\}$. By LTP, we get

$$P(A) = \sum_{i=1}^{4} P(A|B_i)P(B_i)$$

where $P(A|B_1) = 1, P(A|B_2) = P(A|B_3) = 1/3$ and $P(A|B_4) = 1/6$. There are $\binom{8}{3} = 56$ ways to choose three letters and we get $P(B_1) = 1/56, P(B_2) = \binom{3}{2} \cdot 5/56 = 15/56, P(B_3) = 6/56$ and $P(B_4) = 1 - 22/56 = 34/56$. This gives $P(A) = 0.24$.

4. Let $A = \{$ card drawn from second deck is an ace$\}$ and $B_i = \{$ i aces are drawn from the first deck$\}$ for $i = 0, 1, 2$. Then $P(A|B_i) = (4 + i)/54$ and $P(B_i) = \binom{4}{i}\binom{48}{2-i}/\binom{52}{2}$.

a. LTP gives

$$P(A) = \sum_{i=0}^{2} P(A|B_i)P(B_i) = 4/52.$$

Note that this is the same as the probability to draw an ace from an ordinary deck.

b. Bayes' formula gives

$$P(B_0|A) = \frac{P(A|B_0)P(B_0)}{\sum_{i=0}^{2} P(A|B_i)P(B_i)} = \frac{4/54 \cdot \binom{48}{2}/\binom{52}{2}}{4/52} = 0.82.$$

5. The relevant probability is the probability that Mr Smith has the disease given that the diagnosis is positive. Therefore let D be the event that Mr Smith has the disease and C the event that the diagnosis is positive and compute $P(D|C)$. By Bayes' formula we have

$$P(D|C) = \frac{P(C|D)P(D)}{P(C|D)P(D) + P(C|D^c)P(D^c)},$$

where $P(D) = 0.001, P(D^c) = 0.999, P(C|D) = 0.99$ and $P(C|D^c) = 0.01$ since this is the probability that the diagnosis is wrong if the patient does not have the disease. Hence $P(D|C) = 0.99 \cdot 0.001/(0.99 \cdot 0.001 + 0.01 \cdot 0.999) = 0.09$ and Mr Smith is actually quite likely not to have the disease.

To get some intuition for this somewhat suprisingly low probability, think of a population of 100,000 individuals, 100 of which have the disease. If all individuals are tested, we may expect about 99 of those with the disease to be diagnosed as ill. Since the error probability is 0.01, we may also expect about $0.01 \cdot 99,900 = 999$ of the healthy individuals to be diagnosed with the disease (so called "false positives"). Hence there will be an approximate total of 999+99=1098 diagnosed diseases, out of which only 99 individuals, or 9 percent, actually are ill.