Stat 331, Solutions to Homework 4

1a. Since f must be constant and the triangle has area 1/2 we have $f(x, y) = 2$ on the triangle (and 0 otherwise). This gives

$$E[XY] = \int_0^1 \int_0^{1-x} xyf(x, y)dydx = 2 \int_0^1 \int_0^{1-x} xydydx = .. = \frac{1}{12}$$

which gives

$$\rho_{XY} = \frac{E[XY] - E[X]E[Y]}{\sqrt{Var[X]Var[Y]}} = \frac{1/12 - 1/36}{1/18} = -\frac{1}{2}.$$

b. Larger X-values correspond to smaller Y-values so there is a linear trend with negative slope.

2. The circumference C equals $2X + 2Y$ which has a normal distribution with mean $2\mu_1 + 2\mu_2$ and variance $4\sigma_1^2 + 4\sigma_2^2 + 2\rho \cdot 2 \cdot 2\sigma_1\sigma_2$.

a. Here the mean is 30 and the variance 0.328. The probability that a plate is useful is

$$P(29 \leq C \leq 31) = \Phi\left(\frac{31 - 30}{\sqrt{0.328}}\right) - \Phi\left(\frac{29 - 30}{\sqrt{0.328}}\right) = \Phi(1.75) - (1 - \Phi(1.75))$$

$$= 2\Phi(1.75) - 1 = 0.92$$

so the probability that it is useless is 0.08.

b. Given that $X = 5.1$, the plate is useful if Y is between 9.4 and 10.4 inches. The conditional distribution of Y given $X = 5.1$ is $N(10.16, 0.0144)$ which gives

$$P(9.4 \leq Y \leq 10.4|X = 5.1) = \Phi\left(\frac{10.4 - 10.16}{\sqrt{0.0144}}\right) - \Phi\left(\frac{9.4 - 10.16}{\sqrt{0.0144}}\right) = \Phi(2) - \Phi(-6.3) \approx 0.98$$

so the probability that it is useless is ≈ 0.02.

c. Here the mean is 30 and the variance $0.328c$. As above, the probability that a plate is useful is

$$P(29 \leq C \leq 31) = \Phi\left(\frac{31-30}{\sqrt{0.328c}}\right) - \Phi\left(\frac{29-30}{\sqrt{0.328c}}\right) = 2\Phi\left(\frac{1.75}{\sqrt{c}}\right) - 1.$$

To get this at least 0.99, we must have

$$\Phi\left(\frac{1.75}{\sqrt{c}}\right) \geq 0.995$$

which means that the argument of Φ must be at least 2.58, which finally means that c can be no more than $(1.75/2.58)^2 \approx 0.46$.

3. Independence is equivalent to $f(x,y) = f_X(x)f_Y(y)$ and inspection of the pdf's reveals that this holds if and only if $\rho = 0$.

4. First note that the range of T is $(0, \infty)$. Further, we have $T = W + C$ where W and C are independent and hence

$$f_T(x) = \int_0^\infty f_C(x-u)f_W(u)du.$$

Here, $f_C(x-u) = 1$ if $0 \leq x-u \leq 1$ i.e. if $u \geq x-1$ and $u \leq x$. Further, $f_W(u) = e^{-u}$ if $u \geq 0$. This gives two cases:

1. $0 \leq x \leq 1 : f_T(x) = \int_0^x e^{-u}du = 1 - e^{-x}$

2. $x \geq 1 : f_T(x) = \int_{x-1}^x e^{-u}du = e^{-x}(e-1)$.