Stat 331, Solutions to Homework 6

1. The first moment is $\mu_1 = (a+b)/2$ and the second $\mu_2 = E[X^2] = (a^2+ab+b^2)/3$. Solve this system of equations to get a and b expressed in terms of μ_1 and μ_2. Perhaps easier is to notice that the variance is

$$\sigma^2 = \mu_2 - \mu_1^2 = (b-a)^2/12$$

which gives $b - a = \sqrt{12(\mu_2 - \mu_1^2)}$ (remember that $b > a$). Together with the equation $b + a = 2\mu_1$ this gives

$a = \mu_1 - \frac{1}{2}\sqrt{12(\mu_2 - \mu_1^2)}$

$b = \mu_1 + \frac{1}{2}\sqrt{12(\mu_2 - \mu_1^2)}$

which gives method of moments estimators

$\hat{a} = \widehat{\mu}_1 - \frac{1}{2}\sqrt{12(\widehat{\mu}_2 - \widehat{\mu}_1^2)}$

$\hat{b} = \widehat{\mu}_1 + \frac{1}{2}\sqrt{12(\widehat{\mu}_2 - \widehat{\mu}_1^2)}$

2a. We get

$$L(\theta) = \prod_{k=1}^{n} f_\theta(X_k) = \theta^n (\prod_{k=1}^{n} X_k)^{\theta-1}$$

and

$$l(\theta) = \log L(\theta) = n \log\theta + (\theta - 1)\sum_{k=1}^{n}\log X_k.$$

differentiate and set to 0 to obtain

$$\frac{d}{d\theta} l(\theta) = \frac{n}{\theta} + \sum_{k=1}^{n}\log X_k = 0$$

which gives the maximum likelihood estimator

$$\hat{\theta} = -\frac{n}{\sum_{k=1}^{n} \log X_k}$$

(the second derivative is $-n/\theta^2 < 0$.

b. For the method of moments, note that

$$\mu_1 = E[X] = \int_0^1 x f_\theta(x)dx = \theta \int_0^1 x^\theta dx = \frac{\theta}{\theta+1}$$

which gives $\theta = \mu_1/(\mu_1 - 1)$ and method of moments estimator

$$\hat{\theta} = \frac{\hat{\mu}_1}{1-\hat{\mu}_1} = \frac{\bar{X}}{1-\bar{X}}.$$

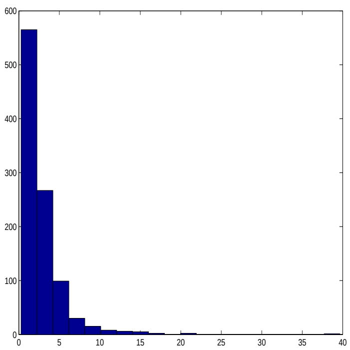
c. The cdf is $F(x) = \int_0^x f(t)dt = x^\theta$ which has inverse $F^{-1}(x) = x^{1/\theta}$. If $U_1, U_2, ...$ are unif[0,1], then $U_1^{1/\theta}, U_2^{1/\theta}, ...$ are observations which have pdf f.

d. Here $F^{-1}(x) = \sqrt{x}$. The following Matlab commands simulate 1000 observations on $\hat{\theta}$, each computed from a sample of size n:

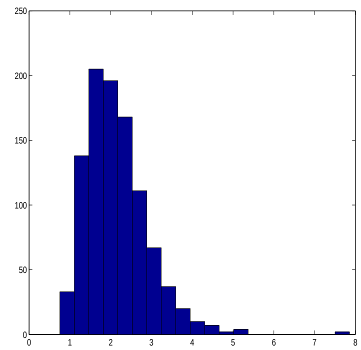
```
u=random('unif',0,1,n,1000);
x=sqrt(u);
t=mean(x)./(1-mean(x));
```

For $n = 3, 10$ and 30, the histograms should look something like this:

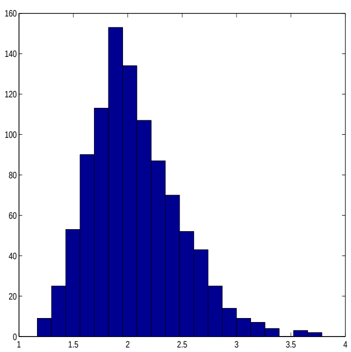
$n = 3$



$n = 10$



$n = 30$



The sample mean is typically higher than 2, but gets closer to 2 as n increases. Also, the sample variance decreases with imcreasing n.

3a. The pdf is

$$f_\theta(x) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{1}{2\theta}x^2}$$

which gives likelihood function

$$L(\theta) = \prod_{k=1}^{n} (2\pi\theta)^{-n/2} e^{-\frac{1}{2\theta}\sum_{k=1}^{n} X_k^2}.$$

Take the logarithm, differentiate with respect to θ and set equal to 0 to obtain the equation

$$-\frac{n}{2\theta} + \frac{1}{2\theta^2} \sum_{k=1}^{n} X_k^2 = 0$$

which gives the MLE $\hat{\theta} = \sum_{k=1}^{n} X_k^2/n$.

Since $E[\hat{\theta}] = \sum_{k=1}^{n} E[X_k^2]/n = \sum_{k=1}^{n}(Var[X_k + (E[X_k])^2)/n = \sum_{k=1}^{n} \theta/n = \theta$, $\hat{\theta}$ is unbiased.

b. The first moment is 0, the second is $\mu_2 = E[X^2] = Var[X_k] + (E[X_k])^2 = \theta$. Since $\theta = \mu_2$, the method of moments estimator is $\hat{\mu}_2 = \sum_{k=1}^{n} X_k^2/n$, the same as the MLE.