Stat 331/Elec 331, Solutions to Final Exam

1a. Since f is non-negative and

$$\int_0^\infty f(x)dx = \frac{1}{2} + \frac{1}{2}\int_1^\infty \frac{1}{x^2}dx = 1$$

f is a possible pdf.

b.

$$E[X] = \int_0^\infty xf(x)dx = \frac{1}{2}\int_0^1 xdx + \frac{1}{2}\int_1^\infty \frac{1}{x}dx = \infty.$$

For $0 \leq x \leq 1$, the cdf is

$$F(x) = \int_0^x f(t)dt = \frac{x}{2}$$

and for $x \geq 1$,

$$F(x) = \int_0^x f(t)dt = \frac{1}{2} + \frac{1}{2}\int_1^x \frac{1}{t^2}dt = \frac{1}{2} + \frac{1}{2}(1 - \frac{1}{x}) = 1 - \frac{1}{2x}.$$

d. The range of Y is $(0, \infty)$. For $0 \leq y \leq 1$, we get

$$F_Y(y) = P(Y \leq y) = P(\frac{1}{X} \leq y) = P(X \geq \frac{1}{y}) = 1 - F_X(\frac{1}{y}) = 1 - (1 - \frac{1}{2(1/y)}) = \frac{y}{2}$$

since $1/y$ is greater than 1. For $y \geq 1$, we get

$$F_Y(y) = 1 - F_X(\frac{1}{y}) = 1 - \frac{1}{2y}$$

since $1/y$ is now bewteen 0 and 1. Hence, Y has the same distribution as X.

2a. Since $C = 2X + 2Y$

$$E[C] = 2E[X] + 2E[Y] = 30$$

$$E[C|X = x] = 2x + 2E[Y|X = x]$$

where

$$E[Y|X=x] = 0.8\frac{\sqrt{0.04}}{\sqrt{0.01}}(x-5) = 1.6x+2$$

and hence

$$E[C|X=x] = 5.2x+4.$$

b. Since $A = XY$ we get

$$E[A] = E[XY] = Cov[X,Y] + E[X]E[Y] = 0.8\sqrt{0.01\cdot 0.04} + 5\cdot 10 = 50.016$$

$$E[A|X=x] = xE[Y|X=x] = x(1.6x+2) = 1.6x^2+2x.$$

c. The probaility that a plate is useful is

$$P(29 \leq C \leq 31) = \Phi\left(\frac{31-30}{\sqrt{0.328}}\right) - \Phi\left(\frac{29-30}{\sqrt{0.328}}\right) = \Phi(1.75) - (1-\Phi(1.75))$$
$$= 2\Phi(1.75) - 1 = 0.92$$

d. Let X=the number of useful plates. Then X has a binomial distribution with $n=10$ and $p = P(\text{useful}) = 0.92$. Hence

$$P(X \geq 9) = P(X=9) + P(X=10) = \binom{10}{9}0.92^9 0.08 + 0.92^{10} = 0.81.$$

3a. The likelihood function is

$$L(a) = \prod_{k=1}^{n} f_a(X_k) = (a+1)^n \prod_{k=1}^{n} X_k^a$$

which gives

$$l(a) = \log L(a) = n\log(a+1) + a\sum_{k=1}^{n}\log X_k$$

and hence

$$\frac{d}{da}l(a) = \frac{n}{a+1} + \sum_{k=1}^{n} \log X_k.$$

Setting this equal to 0 gives the MLE

$$\hat{a} = -\left(\frac{n}{\sum_{k=1}^{n} \log X_k} + 1\right).$$

For the moment estimator, we note that the first moment is

$$\mu_1 = E[X] = \int_0^1 x f(x)dx = (a+1)\int_0^1 x^{a+1}dx = \frac{a+1}{a+2}$$

which gives

$$a = \frac{2\mu - 1}{1-\mu}$$

and hence the moment estimator is

$$\hat{a} = \frac{2\bar{X} - 1}{1 - \bar{X}}.$$

b. If $a = 2$, the pdf is $f(x) = 3x^2$ and the corresponding cdf is

$$F(x) = \int_0^x f(t)dt = \int_0^x 3t^2 dt = x^3, \quad 0 \le x \le 1.$$

The inverse is

$$F^{-1}(x) = x^{1/3}$$

and hence, if $U \sim$ unif $[0,1]$, then $X = U^{1/3}$ has cdf F. If $U = 0.008$, then we get $X = 0.2$.

4a. No, since the times between consecutive A-accidents are $\Gamma(2, \lambda)$ and thus not exponential.

b. For any given week, the probability that it has no accidents is $e^{-\lambda}$, from the Poisson distribution. The random variable N is obtained by counting how many outof 52 weeks that are accident free, and hence $N \sim \text{Bin}(52, e^{-\lambda})$.

c. The first moment is $\mu_1 = E[N] = 52e^{-\lambda}$, and hence

$$\lambda = -\log\left(\frac{\mu_1}{52}\right)$$

and the moment estimator is

$$\widehat{\lambda} = -\log\left(\frac{\bar{X}}{52}\right).$$

In our case, $\bar{X} = N$, and we finally get the moment estimator

$$\widehat{\lambda} = -\log\left(\frac{N}{52}\right).$$

5a. The state space is $\{0, 1, 2, 3\}$ and the rates are

$0 \to 1 : \alpha p$
$0 \to 2 : \alpha(1-p)$
$1 \to 2 : \alpha p$
$1 \to 3 : \alpha(1-p)$
$1 \to 0 : \beta$
$2 \to 3 : \alpha p$
$2 \to 1 : \beta$
$3 \to 2 : \beta$

b. With $p = 1/2$, and $\beta = \alpha$, the balance equations are

State 0: $\alpha\pi_0 = \alpha\pi_1$
State 1: $(2\alpha)\pi_1 = (\alpha/2)\pi_0 + \alpha\pi_2$
State 2: $(3\alpha/2)\pi_2 = (\alpha/2)\pi_0 + (\alpha/2)\pi_1 + \alpha\pi_2$

which together with the condition that $\pi_0 + ... + \pi_3 = 1$ gives stationary distribution

$$\pi_0 = \pi_1 = \frac{4}{19}, \pi_2 = \frac{6}{19}, \pi_3 = \frac{5}{19}.$$

6. The plots in c,d,f,g, and h are from bivariate normal distributions. The correlation coefficient is 0 in f and g (in g, Y has a larger variance than X), -0.99 in c, 0.5 in d, and 0.9 in h. The plot in a has X normal and Y binomial (notice the horisontal lines), the plot in b is uniform on the unit disk, and the plot in e has X normal and Y also normal but centered around the curve $y = x^3$ (notice the shape of the plot).

7. The elements yttrium, erbium, terbium and ytterbium are all named for the village Ytterby outside Stockholm, Sweden.