Solutions to Assignment 1, CAAM/STAT 581

1a. True. If $A = \{a_1, a_2, ...\}$ and $B = \{b_1, b_2, ...\}$ then $A \cup B$ can be enumerated $\{a_1, b_1, a_2, b_2, ...\}$.

b. True. Let $A_j = \{a_{j1}, a_{j2}, ...\}, j = 1, 2, ...$ and enumerate the infinite matrix (a_{ij}) "diagonally" (like we did in class with the rational numbers).

c. True. Let I be the set of irrational numbers. If this is countable, then, since Q is countable, also $R = Q \cup I$ is countable by **a**. But this is a contradiction since R is uncountable.

2a. $\bigcup_{t \in T} A_t = (0, 2), \quad \bigcap_{t \in T} A_t = (0, 1]$.

b. $\bigcup_{t \in T} A_t = (0, 1), \quad \bigcap_{t \in T} A_t = \emptyset$.

c. $\bigcup_{t \in T} A_t = (0, \infty), \quad \bigcap_{t \in T} A_t = \{1\}$.

3. $2^\Omega = \{\emptyset, \Omega\} = \{\emptyset, \{a\}\}, \ 2^{2^\Omega} = \{\emptyset, \{\emptyset\}, \{\Omega\}, 2^\Omega\} = \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\emptyset, \{a\}\}\}$.

Note: a is an element of Ω, $\{a\}$ an element of 2^Ω and $\{\{a\}\}$ an element of 2^{2^Ω}. Also note the difference between $\emptyset$ and $\{\emptyset\}$.

4a. No points belong to infinitely many A_n and hence $\liminf_n A_n = \limsup_n A_n = \emptyset$.

b. The point 0 belongs to all A_n (except A_1 which is empty) and all other points in $(-1, 1)$ belong to infinitely many A_n but are also excluded by infinitely many A_n. Hence $\liminf_n A_n = \{0\}, \limsup_n A_n = (-1, 1)$

c. All points in $[0, \infty)$ belong to all but finitely many A_n and no other points belong to infinitely many A_n. Hence $\liminf_n A_n = \limsup_n A_n = [0, \infty)$.

d. First note that the points $(0, 1)$ and $(0, -1)$ do not belong to any A_n. All the points in the *closed* unit circle disc belong to infinetly many A_n but only the points in the *open* disc belong to A_n eventually. Hence $\liminf_n A_n$ is the open unit disc and $\limsup_n A_n$ is the closed unit disc except the points $(0, 1)$ and $(0, -1)$. It is useful to sketch a figure to realize how the A_n "settle in" towards the unit disk.