Assignment 2, CAAM/STAT 581, due September 19

1. Consider an experiment where you generate an infinite sequence of integers such that in the nth step you choose one of the numbers $\{1, 2, ..., n\}$ at random, $n = 1, 2, ...$

a. Suggest a sample space Ω for this experiment.

b. Let A be the event that the sequence ends with an infinite string of 1's. Use Fatou's Lemma to show that $P(A) = 0$.

2. Let $\mathcal{B}$ be a σ-field. Which of the following statements are true? Give proofs or counterexamples.

a. $A \in \mathcal{B}, B \notin \mathcal{B} \Rightarrow A \cup B \notin \mathcal{B}$

b. $A \in \mathcal{B}, B \notin \mathcal{B}, A \cap B = \emptyset \Rightarrow A \cup B \notin \mathcal{B}$

c. $A \notin \mathcal{B}, B \notin \mathcal{B} \Rightarrow A \cup B \notin \mathcal{B}$

3. Consider a class $\mathcal{H}$ of subsets of Ω which is closed under complements and finite, *disjoint* unions. Consider the following argument: Take two sets $A, B \in \mathcal{H}$, not necessarily disjoint. Now, any union can be represented as a disjoint union: Let $C = A$ and $D = B \setminus A$, then C and D are disjoint and $A \cup B = C \cup D$. Since $\mathcal{H}$ is closed under disjoint unions, $C \cup D$ and hence $A \cup B$ belongs to $\mathcal{H}$. But this shows that $\mathcal{H}$ is in fact closed under arbitrary finite unions and is thus a field.

Give an example to show that the argument is false. Where is the error?

4. Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-fields. Show that the union $\mathcal{A} \cup \mathcal{B}$ is not necessarily a σ-field.

5. Let $\mathcal{F}$ be a class of subsets of Ω such that

(i) $\Omega \in \mathcal{F}$

(ii) $A, B \in \mathcal{F} \Rightarrow A \cap B^c \in \mathcal{F}$.

Show that $\mathcal{F}$ is a field.

6. Let $\mathcal{F}_1, \mathcal{F}_2, ...$ be classes of subsets of Ω such that $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq ...$ and let $\mathcal{F} = \cup_{n=1}^{\infty} \mathcal{F}_n$. Show that

a. If the $\mathcal{F}_n$ are fields, then $\mathcal{F}$ is also a field.

b. If the $\mathcal{F}_n$ are σ-fields, then $\mathcal{F}$ is not necessarily a σ-field.