Solutions to Assignment 2, CAAM/STAT 581

1a. Since an outcome is an infinite sequence of integers where the kth integer is bewteen 1 and k, Ω can be taken as the set of such sequences. Hence $\Omega = \{\omega = (\omega_1, \omega_2, ...) : \omega_k \in \{1, ..., k\}, k = 1, 2, ...\}$.

b. Let $A_n = \{$ the nth number is $1\}$. Then A_n is a subset of Ω since $A_n = \{\omega \in \Omega : \omega_n = 1\}$. By the description of the experiment, $P(A_n) = 1/n$.

Now note that any outcome in A has the property that it ends with an infinite sequence of 1's. This means that it has 1's in *all but finitely many positions*, i.e. $A = \liminf_n A_n$. By the first part of Fatou's lemma, we get $P(A) = P(\liminf_n A_n) \leq \liminf_n P(A_n) = \liminf_n(1/n) = 0$ and hence $P(A) = 0$.

Note that A has probability 0 but is not empty; in fact, there are infinitely many outcomes in A.

2a. False. Take any $B \notin \mathcal{B}$ and $A = \Omega$. Then $A \cup B = \Omega \in \mathcal{B}$.

b. True. Assume $A \cup B \in \mathcal{F}$. Then $B = (A \cup B) \cap A^c \in \mathcal{B}$, a contradiction.

c. False. Take any $A \notin \mathcal{B}$ and $B = A^c$. Then $A \cup B = \Omega \in \mathcal{B}$.

3. Let $\Omega = \{1, 2, 3, 4\}$ and $\mathcal{H} = \{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \Omega\}$. This is closed under complements and disjoint unions but not arbitrary unions since $\{1, 2\} \in \mathcal{H}$ and $\{1, 3\} \in \mathcal{H}$ but $\{1, 2, 3\} \notin \mathcal{H}$.

The error is that if A and B are in $\mathcal{H}$, $B \setminus A$ need not be. In the example, with $A = \{1, 2\}$ and $B = \{1, 3\}$, we have $B \setminus A = \{3\}$ which is not in $\mathcal{H}$.

4. The union $\mathcal{A} \cup \mathcal{B}$ need not be closed under unions and may thus fail to be a σ-field. For example, let $\Omega = \{1, 2, 3\}, \mathcal{A} = \{\emptyset, \{1\}, \{2, 3\}, \Omega\}$ and $\mathcal{B} = \{\emptyset, \{2\}, \{1, 3\}, \Omega\}$. Then $\mathcal{A} \cup \mathcal{B} = \{\emptyset, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, \Omega\}$

which is not a σ-field. For example, $\{1\} \in \mathcal{A} \cup \mathcal{B}$ and $\{2\} \in \mathcal{A} \cup \mathcal{B}$ but $\{1\} \cup \{2\} = \{1, 2\} \notin \mathcal{A} \cup \mathcal{B}$.

5. $\Omega \in \mathcal{F}$ is given. By choosing $A = \Omega$, we see that $\mathcal{F}$ is closed under complement. Finally, take $A, B \in \mathcal{F}$. Then $A^c \in \mathcal{F}$ and hence $A^c \cap B^c \in \mathcal{F}$ and since $\mathcal{F}$ is closed under complement, $A \cup B = (A^c \cap B^c)^c \in \mathcal{F}$ and $\mathcal{F}$ is a field.

6a. Since Ω is in each of the $\mathcal{F}_n$, $\Omega \in \mathcal{F}$. Take $A \in \mathcal{F}$. Then $A \in \mathcal{F}_n$ for some n and hence $A^c \in \mathcal{F}_n$ for this n so that $A^c \in \mathcal{F}$. Finally, take $A, B \in \mathcal{F}$. Then $A \in F_n$ for some n and $B \in \mathcal{F}_m$ for some m. Let $k = \max(m, n)$ and note that $A, B \in \mathcal{F}_k \Rightarrow A \cup B \in \mathcal{F}_k \Rightarrow A \cup B \in \mathcal{F}$. Hence $\mathcal{F}$ is a field.

b. Countable additivity may fail. For example, let $\Omega = \{1, 2, ...\}$ and let $\mathcal{F}_k$ be the σ-field generated by the class $\{\{1\}, \{2\}, ..., \{k\}\}$ (so that $\mathcal{F}_1 = \{\emptyset, \{1\}, \{2, 3, ...\}, \Omega\}, \mathcal{F}_2 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3, ...\}, \{1, 3, ...\}, \{3, 4, ...\}, \Omega\}$ and so on). Now let $A_k = \{2k - 1\},\ k = 1, 2, ...$ and $A = \bigcup_k A_k$. Then $A_k \in \mathcal{F}_{2k-1}$ so that $A_k \in \mathcal{F}$ for each k. But $A = \bigcup_{k=1}^{\infty} \{k\} = \{1, 3, 5, ...\}$ which is not in $\mathcal{F}_k$ for any k and hence not in $\mathcal{F}$.

It can be shown that if the $\mathcal{F}_n$ are such that $\mathcal{F}_n$ is always a *proper* subset of $\mathcal{F}_{n+1}$, then $\mathcal{F}$ can *never* be a σ-field. Clearly, this is not true without the proper subset requirement; if for example $\mathcal{F}_1 = \mathcal{F}_2 = ...$, then $\mathcal{F}$ is a σ-field.