Assignment 3, CAAM/STAT 581, due October 1

1. Let Ω be the following square in $R^2 : \Omega = (-1, 1) \times (-1, 1) = \{(x, y) : -1 < x < 1, -1 < y < 1\}$. Define an open rectangle R_{abcd} to be the set $(a, b) \times (c, d) = \{(x, y) : a < x < b, c < y < d\}$ where a, b, c, d are in $[-1, 1]$. Also, define $\emptyset$ to be an open rectangle. Let $\mathcal{C}$ be the class of open rectangles and let $\mathcal{B}$ be the σ-field generated by $\mathcal{C}$.

a. Show that $\mathcal{B}$ contains the following subsets of Ω:

(i) the closed rectangles, $[a, b] \times [c, d]$, $a, b, c, d \in (-1, 1)$.

(ii) the vertical line $\{(x, y) \in \Omega : x = 0, -1 < y < 1\}$

(iii) the diagonal line $\{(x, y) \in \Omega : x = y\}$

b. Let μ be a function, $\mu : \mathcal{B} \to [0, \infty)$ such that

(i) If $B_1, B_2, ...$ are disjoint sets in $\mathcal{B}$, then $\mu(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n)$

(ii) If R_{abcd} is an open rectangle, then $\mu(R_{abcd}) = (b - a)(d - c)$ and $\mu(\emptyset) = 0$.

Show that for any set $B \in \mathcal{B}$, $\mu(B)$ is the area of B (with the convention that if $A_1, A_2, ...$ are disjoint sets in R^2, then the area of $\bigcup_{k=1}^{\infty} A_k$ equals $\sum_{k=1}^{\infty}$ (area of A_k)).

2. Let $\mathcal{S}$ be a semi-algebra and let $\mathcal{A}(\mathcal{S})$ be the field generated by $\mathcal{S}$. Show that $\sigma(\mathcal{S}) = \sigma(\mathcal{A}(\mathcal{S}))$ (easiest by showing that each is included in the other).

3. Let $\Omega = \{1, 2, 3, 4\}$ and let $\mathcal{S} = \{\emptyset, \{1\}, \{2\}, \{3, 4\}, \Omega\}$. Let P be a probability measure on $\mathcal{S}$ such that $\{1\}, \{2\}$ and $\{3, 4\}$ all have probability 1/3. Define Π^* and $\mathcal{D}$ as in the proof of Theorem 2.4.2.

a. Show that $\mathcal{S}$ is a semi-algebra and find $\mathcal{A}(\mathcal{S})$ and $\sigma(\mathcal{S})$.

b. Find $\mathcal{D}$.

4. Show that the Borel σ-field on $(0, 1]$ is not closed under uncountable unions.

5. Show that the point $3/4$ is in the Cantor set.