Solutions to Assignment 3, CAAM/STAT 581

1a.

(i) We get a closed rectangle from open rectangles by

$$[a, b] \times [c, d] = \bigcap_n (a - \frac{1}{n}, b + \frac{1}{n}) \times (c - \frac{1}{n}, d + \frac{1}{n})$$

where of course n has to be large enough for the open rectangles to be in Ω.

(ii)

$$\{x\} \times (-1, 1) = \bigcap_n (x - \frac{1}{n}, x + \frac{1}{n}) \times (-1, 1).$$

(iii) First divide Ω into four equally sized squares in the four quadrants. Then divide the closed "northeast" and "southwest" quadrants in the same way and proceed indefinitely. The line $\{(x, y) : x = y\}$ is the intersection of all the closed squares on that diagonal. Remember that you are *not* allowed to use *uncountable* unions and intersections.

b. Let $A(B)$ denote the area of a set B and let $\mathcal{L}$ be the class of sets in $\mathcal{B}$ for which $\mu(B) = A(B)$. We show that $\mathcal{L}$ is a λ-system:

($\bullet$) $\Omega \in \mathcal{L}$ since $\mu(\Omega) = \mu(R_{-1,1,-1,1} = 4 =$ the area of Ω.

($\bullet$) Since $\emptyset$ is an open rectangle ($a = b, c = d$ in the definition), (ii) yields $\mu(\emptyset) = 0$. Further, (i) yields $4 = \mu(\Omega) = \mu(B \cup B^c \cup \emptyset \cup \emptyset ...) = \mu(B) + \mu(B^c) + \mu(\emptyset) + \mu(\emptyset) + ... = \mu(B) + \mu(B^c) + 0 + 0 + ...$ which gives $\mu(B^c) = 4 - \mu(B)$ for any $B \in \mathcal{B}$. But also $A(B^c) = 4 - A(B)$ so $\mathcal{L}$ is closed under complement.

($\bullet$) Take a disjoint sequence $B_1, B_2, ...$ in $\mathcal{L}$ and let $B = \cup_n B_n$. By (i), $\mu(B) = \sum_n \mu(B_n) = \sum_n A(B_n) = A(B)$ so $\mathcal{L}$ is closed under disjoint unions.

The class $\mathcal{C}$ of open rectangels is a π-system since the intersection of two open rectangles is again an open rectangle. Also, by (ii), $\mu(C) = A(C)$ for sets $C \in \mathcal{C}$. Hence $\mathcal{L}$ contains $\mathcal{C}$ and by the $\pi - \lambda$ theorem, $\mathcal{L}$ contains $\mathcal{B} = \sigma(\mathcal{C})$ i.e. $\mu(B) = A(B)$ for any set $B \in \mathcal{B}$.

2. We need the following observation: If $\mathcal{C}$ is any class of sets and $\mathcal{B}$ is any σ-field such that $\mathcal{C} \subseteq \mathcal{B}$, then also $\sigma(\mathcal{C}) \subseteq \mathcal{B}$. Remember that $\sigma(\mathcal{C})$ is the *smallest* σ-field containing $\mathcal{C}$.

Now, $\mathcal{S} \subseteq \mathcal{A}(\mathcal{S}) \subseteq \sigma(\mathcal{A}(\mathcal{S}))$ and hence $\sigma(\mathcal{S}) \subseteq \sigma(\mathcal{A}(\mathcal{S}))$. To show the other inclusion, note that $\mathcal{A}(\mathcal{S}) \subseteq \sigma(\mathcal{S})$ (since $\sigma(\mathcal{S})$ is also a field and $\mathcal{A}(\mathcal{S})$ is minimal) and hence $\sigma(\mathcal{A}(\mathcal{S})) \subseteq \sigma(\mathcal{S})$.

Note that this gives $\sigma(\mathcal{S}) = \sigma(\mathcal{A}(\mathcal{S}))$ for any class $\mathcal{S}$, not only for semi-algebras.

3a. Clearly, $\emptyset \in \mathcal{S}$, $\Omega \in \mathcal{S}$. Further, any intersection of sets in $\mathcal{S}$ gives a set in $\mathcal{S}$. Finally, any complement is a disjoint union of sets in $\mathcal{S}$, for example, $\{1\}^c = \{2,3,4\} = \{2\} \cup \{3,4\}$. The field generated by $\mathcal{S}$ is $\mathcal{A}(\mathcal{S}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3,4\}, \{2,3,4\}, \Omega\}$, the class of finite, disjoint unions of sets in $\mathcal{S}$.

b. All sets in $\mathcal{A}(\mathcal{S})$ are in $\mathcal{D}$. The set $\{3\}$ is not in $\mathcal{D}$ since $\Pi^*(\{3\}) = P(\{3,4\}) = 1/3$ and $\Pi^*(\{3\}^c) = \Pi^*(\{1,2,4\} = P(\Omega\}) = 1$ so $\Pi^*(\{3\}) + \Pi^*(\{3\}^c) = 4/3$. But then no other sets outside $\mathcal{A}(\mathcal{S})$ can be in $\mathcal{D}$ either (for example, if $\{2,3\} \in \mathcal{D}$, then $\{3\} = \{2,3\} \cap \{3,4\}$ would be in $\mathcal{D}$ also). Hence $\mathcal{D} = \mathcal{A}(\mathcal{S})$.

4. Let A be the non-measurable set we constructed in class. Since all countable sets are Borel sets, A must be uncountable. Now, any set is the union of its points so $A = \bigcup_{x \in A}\{x\}$ where each member of the union is a Borel set but the union itself is not.

5. Since

$$2 \cdot 1/3 + 2 \cdot (1/3)^3 = ... = 2\sum_{n=0}^{\infty}(1/3)^{2n+1} =$$

$$\frac{2}{3}\sum_{n=0}^{\infty}(\frac{1}{9})^n = \frac{2}{3}\frac{1}{1-\frac{1}{9}} = \frac{3}{4}$$

the base-3 expansion of $3/4$ is 0.2020... which has no 1's and thus belongs to the Cantor set.