Solutions to Assignment 4, CAAM/STAT 581

1a. (i) Take a Borel set B. Then $h^{-1}(B) = B$ so h generates the Borel σ-field.

(ii). Let $B = (a, b]$. Then $h^{-1}(B) = \{x : a < h(x) \leq b\} = (\sqrt{a}, \sqrt{b}]$ Since we can obtain any interval as an inverse image under h, h generates the Borel σ-field.

(iii). $h^{-1}(\{1\}) = Q, h^{-1}(\{0\}) = Q^c$ and except for $\emptyset$ and $[0, \infty)$, no other sets can be obtained as inverse images under h. Hence $\sigma(h) = \{\emptyset, Q, Q^c, [0, \infty)\}$.

b. Let A be the non-measurable set constructed in class and let h be the indicator function I_A. Then $\{1\}$ is a Borel set but $h^{-1}(\{1\}) = A$ is not, so h is not measurable.

2a. Since a singleton set is countable, we have $\mathcal{C} \subseteq \mathcal{A}$, and hence $\sigma(\mathcal{C}) \subseteq \mathcal{A}$ by minimality. Conversely, take a set $B \in \mathcal{A}$. If B is countable, then $B = \cup_{x \in B}\{x\}$ and if B^c is countable, then $B = (\cup_{x \in B^c}\{x\})^c$. In either case, B can be obtained by countable set operations from sets in $\mathcal{C}$ and hence $B \in \sigma(\mathcal{C})$ means that $\mathcal{A} \subseteq \sigma(\mathcal{C})$.

b. $[0, 1]$ is a Borel set which is uncountable and has an uncountable complement and is thus not in $\mathcal{A}$.

c. (i) Not measurable since $h^{-1}([0, 1]) = [0, 1] \notin \mathcal{A}$.

(ii) Measurable. Take a Borel set B. If $0 \in B$, then $h^{-1}(B) = Q^c \bigcup (Q \cap B)$ which has a countable complement. If $0 \notin B$, then $h^{-1}(B) = Q \cap B$ which is countable.

(iii) Not measurable since $h^{-1}([0, 1]) = [0, 1] \bigcup (Q \bigcap (0, \infty))$ which is uncountable and has an uncountable complement.

(iv) Not measurable since $h^{-1}(\{1\}) = [0, 1] \notin \mathcal{A}$.