Solutions to Assignment 5, CAAM/STAT 581

1a. $P(X_n > \log n) = e^{-\log n} = 1/n$ and since $\sum_n 1/n = \infty$, $P(X_n > \log n$ i.o. $) = 1$ by the second B-C lemma.

b. $P(X_n > n) = e^{-n}$ and since $\sum_n e^{-n} = \sum_n (e^{-1})^n < \infty$ $(e^{-1} < 1)$, the first B-C lemma gives that $P(X_n > n$ i.o. $) = 0$.

2. Let $X_1, X_2, ...$ be such that $X_k = 1/k^2$ or 0 with equal probabilities. The event $\{\sum_k X_k < 1\}$ is symmetric but has probability 1/2 (the sum is < 1 if and only if $X_1 = 0$).

3. Let $X(\omega) = 2\omega - 1$. Then, $P(a \leq X \leq b) = P(\{\omega \in [0,1] : X(\omega) \in [a,b]) = P(\{\omega \in [0,1] : 2\omega - 1 \in [a,b]) = P(\{\omega \in [0,1] : \omega \in [(a+1)/2, (b+1)/2] = (b-a)/2$ since P is Lebesgue measure.

Note that we can obtain any uniform distribution with this type of construction. To get uniform distribution on $[a,b]$, simply define $X(\omega) = a + (b-a)\omega$. Later in the course we will see that we can get *any* distribution by using this probability space and smiilar ideas.

First divide X into X^+ and X^-. By symmetry, $E[X^+] = E[X^-]$, and by choosing appropriate sequences of simple functions, it can be shown that $E[X^+] = E[X^-] = 1/4$, and hence $E[X] = 0$.

4. a.. Let

$$X_n = \begin{cases} 0 & \text{with probability } 1 - 1/n^2 \\ n^2 & \text{with probability } 1/n^2 \end{cases}$$

b.. Let

$$X_n = \begin{cases} 0 & \text{with probability } 1 - 1/n^2 \\ n^2 - 1 & \text{with probability } 1/n^2 \end{cases}$$

c. Let

$$X_n = \begin{cases} n & \text{with probability } 1 - 1/n^2 \\ -n^4 & \text{with probability } 1/n^2 \end{cases}$$

d. Let

$$X_n = \begin{cases} -1 & \text{with probability } 1/2 \\ 1 & \text{with probability } 1/2 \end{cases}$$