Assignment 6, CAAM/STAT 581, due November 19

1. For the following functions, determine if the integral $\int_{-\infty}^{\infty} f(x)dx$ exists and if it does, compute its value.

a. $$f(x) = \begin{cases} -1/x^2 & \text{if } x \leq -1 \\ x & \text{if } -1 \leq x \leq 1 \\ 1/x^2 & \text{if } x \geq 1 \end{cases}$$

b. $$f(x) = \begin{cases} -1/x^2 & \text{if } x \leq -1 \\ x & \text{if } -1 \leq x \leq 1 \\ 1/x & \text{if } x \geq 1 \end{cases}$$

c. $$f(x) = \begin{cases} 1/x & \text{if } x \leq -1 \\ x & \text{if } -1 \leq x \leq 1 \\ 1/x & \text{if } x \geq 1 \end{cases}$$

2. Compute the limit $\lim_{n\to\infty} \int_0^\infty f_n(x)dx$ for the following functions.

a.

$$f_n(x) = \frac{n\sin(\frac{x}{n})}{x(1+x^2)}$$

b.

$$f_n(x) = \begin{cases} 1/n & \text{if } x \geq n \\ 0 & \text{if } x < n \end{cases}$$

3. If X is a discrete random variable with range $\{1, 2, ...\}$, then it is a well known result that

$$E[X] = \sum_{n=1}^{\infty} P(X \geq n).$$

Show that this result follows from Corollary 5.3.1. (write X as a sum of indicators).

4. Let $X_1, X_2, \ldots$ be random variables such that $X_n \geq 0$ and $E[X_n] \leq K < \infty$ for all n and $X_n \to X$ a.s. Is it then true that $E[X] \leq K$? What if the condition $X_n \geq 0$ is replaced by the weaker condition $E[X_n] \geq 0$? Proofs or counterexamples.

5. Let $(\Omega, \mathcal{B}, P)$ be a probability space, let $C \in \mathcal{B}$ and let X be a random variable which is independent of C (meaning that $P(A \cap C) = P(A)P(C)$ for all sets $A \in \sigma(X)$). Show that $E[X; C] = E[X]P(C)$.