Solutions to Assignment 6, CAAM/STAT 581

1a. $\int f^+(x)dx = 1/2 + \int_1^\infty 1/x^2\ dx = 3/2$. By symmetry, $\int f^-(x)dx = 3/2$ and hence $\int f(x)dx = 0$.

b. $\int f^+(x)dx = 1/2 + \int_1^\infty 1/x\ dx = \infty$, $\int f^-(x)dx = 3/2$ and hence $\int f(x)dx = \infty$.

c. $\int f^+(x)dx = 1/2 + \int_1^\infty 1/x\ dx = \infty$. By symmetry, $\int f^-(x)dx = \infty$ so $\int_{-\infty}^\infty f(x)dx$ does not exist.

2a. Since $|\sin(t)/t| \le 1$ and $\sin(t)/t \to 1$ as $t \to 0$, we have $f_n(x) \to 1/(1+x^2)$ and $|f_n(x)| \le 1/(1+x^2)$ which is integrable on $(0,\infty)$. Hence DCT applies and we get $\lim_n \int_0^\infty f_n(x)dx = \int_0^\infty 1/(1+x^2)\ dx = \pi/2$.

b. Here neither DCT nor MCT applies but direct computation gives that $\int_0^\infty f_n(x)dx = \infty$ for all n and hence the limit is ∞ (note that $f_n \to 0$ though).

3. Note that $X = \sum_{n=1}^\infty I_{\{X\ge n\}}$ (if $X = k$, then the sum contains exactly k 1's). Since the summands are non-negative, Corollary 5.3.1 applies to give $E[X] = E[\sum_{n=1}^\infty I_{\{X\ge n\}}] = \sum_{n=1}^\infty E[I_{\{X\ge n\}}] = \sum_{n=1}^\infty P(X \ge n)$.

If we want to spell out the arguments, we get $X(\omega) = \sum_n I_{X^{-1}([n,\infty))}(\omega)$.

4. By Fatou's Lemma, $E[X] \le \liminf_n E[X_n] \le K$ so it is true. With the weaker condition $E[X_n] \ge 0$ it is not true anymore; a counterexample is to let

$$X_n = \begin{cases} 1 & \text{with probability } 1 - 1/n^2 \\ 1 - n^2 & \text{with probability } 1/n^2 \end{cases}$$

Then $E[X_n] \equiv 0$ and $X_n \to 1$ a.s. by the first Borel-Cantelli so the statement fails with $K = 0$ and $X \equiv 1$.

5. Start with indicators.

1. $X = I_A$. Then $E[X;C] = E[I_A I_C] = P(A \cap C) = P(A)P(C) = E[X]P(C)$

2. X simple, $X = \sum_{i=1}^k a_i I_{A_i}$. Then $E[X;C] = \sum_{i=1}^k a_i E[I_{A_i};C] = ... = E[X]P(C)$.

3. $X \geq 0$. Take simple, non-negative $X_n \uparrow X$ and note that $X_n I_C \uparrow X I_C$ and hence, by 2, $E[X;C] = \lim_n E[X_n;C] = \lim_n E[X_n]P(C) = E[X]P(C)$.

4. X general, write $X = X^+ - X^-$. Then, $(XI_C)^+ = X^+ I_C$ and $(XI_C)^- = X^- I_C$ and by 3, $E[X;C] = E[X^+;C] - E[X^-;C] = E[X^+]P(C) - E[X^-]P(C) = E[X]P(C)$.