Final Exam, STAT 581, Fall 2003

1. Let $\mathcal{B}$ be the Borel sets on R and define the class $\mathcal{F} = \{A \in \mathcal{B} : A$ is finite or A^c is finite$\}$ (where "finite" as usual means having a finite number of points). Let $\mu : \mathcal{F} \to R$ be a function such that $\mu(A) = 0$ if A is finite and $\mu(A) = 2$ if A^c is finite.

a. Show that $\mathcal{F}$ is a field but not a σ-field. What is the σ-field generated by $\mathcal{F}$?

b. Show that μ is a measure on $(R, \mathcal{F})$.

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2a. State the first and second Borel-Cantelli lemmas.

b. Use Borel-Cantelli to construct examples of sequences of independent random variables $X_1, X_2, ...$ such that

(i) $X_n \to 0 \;\; a.s.$ and $E[X_n] \equiv 1$.

(ii) X_n does not converge a.s., $E[X_n] < 1$ for all n and $E[X_n] \to 1$.

(iii) $X_n \to \infty \;\; a.s.$ and $E[X_n] \equiv 0$.

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3. Find $\lim_{n\to\infty} \int_1^\infty f_n(x)dx$ for the following functions (note the limits of the integral):

(i) $f_n(x) = I_{[n,n+1]}(x)$

(ii) $f_n(x) = \sin(nx)e^{-nx}x^{-2}$

(iii) $f_n(x) = xn^{-1}$

(iv) $f_n(x) = (-1)^n xn^{-2}$.

(v) $f_n(x) = (1 + nx^2)(1 + x^2)^{-n}$

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4a. State Fatou's lemma for expectations.

b. Fatou's lemma for probabilities states that if $A_1, A_2, ...$ is a sequence of events, then $P(\liminf_n A_n) \leq \liminf_n P(A_n)$. Show that this follows from Fatou's lemma for expectations.

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5. Let $(\Omega, \mathcal{B}, \mu)$ be a measure space. Fix a set $B \in \mathcal{B}$ and define $\nu(A) = \mu(A \cap B)$ for $A \in \mathcal{B}$.

a. Show that ν is a measure on $(\Omega, \mathcal{B})$.

b. Let $g : \Omega \to R$ be a function such that $\int_B g d\mu$ exists. Show that

$$\int_B g d\mu = \int_\Omega g d\nu.$$

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6a. Let X be a non-negative random variable. Show that

$$E[X] = \int_0^\infty (1 - F(x)) dx.$$

Hint: Recall that if X has range $\{1, 2, ...\}$, then $E[X] = \sum_n P(X \geq n)$ which can be viewed as changing the order of integration. The same type of method applies here.

b. Let $X_1, X_2, ...,$ be i.i.d. non-negative random variables with infinite expectation. Show that $P(X_n > n \text{ i.o. }) = 1$ (you may use the result in **a** even if you did not manage to prove it).