Solutions to Assignment 1, STAT 582

1a. Take $\epsilon > 0$. Since

$$\sum_{n=1}^{\infty} P(|X_n - X| > \epsilon) = \sum_{n=1}^{\infty} P(Y_n > \epsilon) = \sum_{n=1}^{\infty} e^{-n\epsilon} < \infty,$$

$X_n \to X$ a.s.

b. Note that $X_n \neq X$ if and only if $Y_n \neq 0$ which has probability 1 for all n. Hence, the probability is 1.

2a. Let

$$X_n = \begin{cases} n^2 & \text{with probability } 1/n^2 \\ 0 & \text{with probability } 1 - 1/n^2 \end{cases}$$

Then $X_n \to 0$ by the first Borel-Cantelli Lemma, $E[|X_n - 0|] = E[X_n] = 1/n \to 0$ but $E[X_n^2] \equiv 1 \not\to 0$.

b. Let

$$X_n = \begin{cases} 2^n & \text{with probability } 1/n^2 \\ 0 & \text{with probability } 1 - 1/n^2 \end{cases}$$

By the first B-C, $X_n \to 0$ a.s. Since $E[X_n^p] = 2^{np}/n^2 \to \infty$, X_n does not converge to 0 in L_p for any p.

3. By first Borel-Cantelli, $X_n \to 0$ a.s. However, $E[|X_n - 0|] = E[|X_n|] \equiv 1$ so there is no convergence in L_1.

4. Start with an arbitrary subsequence n_k. Then there is a further subsequence $n_k^{'}$ such that $X_{n_k^{'}} \to X$ a.s. Since also $X_{n_k^{'}} \xrightarrow{P} X$, there is a subsequence of $n_k^{'}$, call it $n_k^{''}$ such that $Y_{n_k^{''}} \to Y$ a.s. By what is known about a.s. convergence, we get $X_{n_k^{''}} + Y_{n_k^{''}} \to X + Y$ a.s. But since n_k was arbitrary, this means that every subsequence $\{n_k\}$has a further subsequence $\{n_k^{''}$ where there is a.s. convergence, and hence $X_n + Y_n \xrightarrow{P} X + Y$ Note how we need to do this in two steps, first for the X's and then for the Y's.

5a. The cdf, $F(x)$, is 0 for $x < c$ and 1 for $x \geq c$. Convergence in distribution fo c means that $F_n(x) \to 0$ for $x < c$ and $\to 1$ for $x > c$ (no requirements on the discontinuity point $x = c$).

b. Take $\epsilon > 0$ to obtain

$$P(|X_n - c| > \epsilon) = 1 - P(|X_n - c| \leq \epsilon)$$

where

$$P(|X_n - c| \leq \epsilon) = P(c - \epsilon \leq X_n \leq c + \epsilon)$$
$$\geq P(c - \epsilon < X_n \leq c + \epsilon) = F_n(c + \epsilon) - F_n(c - \epsilon) \to 1.$$

Hence $P(|X_n - c| > \epsilon) \to 0$ i.e. $X_n \xrightarrow{P} c$.