Solutions to Assignment 2, STAT 582

1a. Since $E[S_n/n] = \mu$ and $Var[S_n/n] = \sigma^2/n$, Chebyshev's inequality (with $p = 2$) gives

$$P(|S_n/n - \mu| > \epsilon) \le \frac{\sigma^2}{n\epsilon^2} \to 0.$$

Alternatively, use Theorem 7.2.1.

b. Let

$$X_j = \begin{cases} -j & \text{with probability } 1/2 \\ j & \text{with probability } 1/2 \end{cases}$$

Then $E[X_j] = 0$ for all j but since $E[X_j^2; |X_j| \le n] = E[X_j^2] = j^2$, condition (ii) in Theorem 7.2.1 fails and hence S_n/n does not converge in probability to 0.

2. Use the version of Theorem 7.2.1 which has "b_n" instead of "n" and choose $b_n = n^2$. Now check the conditions:

(i). Since $|X_j| \le j$, we get $\sum_{j=1}^n P(|X_j| > n^2) = 0$ so this condition holds trivially.

(ii). Again since $|X_j| \le j$, we get $E[X_j^2 : |X_j| \le n^2] = E[X_j^2] = j^2/3$ and hence

$$\frac{1}{n^4}\sum_{j=1}^n E[X_j^2; |X_j| \le n^2] = \frac{1}{3n^4}\sum_{j=1}^n j^2 \to 0$$

since $\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}$.

Finally, to identify the limit, note that

$$a_n = \sum_{j=1}^n E[X_j; |X_j| \le n^2 = \sum_{j=1}^n E[X_j] = \sum_{j=1}^n \frac{j}{2} = \frac{n(n+1)}{4}$$

so that $a_n/n^2 \to 1/4$ and we conclude

$$\frac{S_n}{n^2} \to \frac{1}{4}.$$

3a. $c_j = 2/j^2$.

b. Again, use the general version of Theorem 7.2.1 with $b_n = \log n$. Condition (i) becomes

$$\sum_{j=1}^{n} P(X_j > \log n) = \sum_{j=1}^{n} \frac{2}{j^2} \int_{\log n}^{\infty} \frac{dx}{x^3} = \frac{2}{\log^2 n} \sum_{j=1}^{n} \frac{1}{j^2} \to 0$$

and condition (ii)

$$\begin{aligned}\frac{1}{\log^2 n} \sum_{j=1}^{n} E[X_j^2; X_j \le \log n] &= \frac{1}{\log^2 n} \sum_{j=1}^{n} \frac{2}{j^2} \int_{1/j}^{\log n} \frac{dx}{x} \\ &= \frac{1}{\log^2 n} \sum_{j=1}^{n} \frac{2}{j^2} (\log \log n + \log j)\end{aligned}$$

which goes to 0 as $n \to \infty$ since $\log^2 n$ goes to ∞ faster than $\log \log n$ and $\sum_{j=1}^{n} \log j / j^2 < \infty$. Hence we have $(S_n - a_n)/\log n \xrightarrow{P} 0$. Finally

$$E[X_j; |X_j| \le \log n] = \frac{2}{j^2} \int_{1/j}^{\log n} \frac{dx}{x^2} = \frac{2}{j^2}(j - \frac{1}{\log n}) = \frac{2}{j} - \frac{2}{j^2 \log n}$$

so that

$$a_n = \sum_{j=1}^{n} \frac{2}{j} - \sum_{j=1}^{n} \frac{2}{j^2 \log n}$$

which has $a_n/\log n \to 2$ as $n \to \infty$ and hence $S_n/\log n \xrightarrow{P} 2$.