Assignment 3, STAT 582, due February 27

1a. In the random signs problem, suppose that we choose '+' with probability p and '-' with probability $1-p$. Show that the sum $\sum_{j=1}^{\infty}(X_j/j)$ equals ∞ a.s. if $p > 1/2$ and $-\infty$ a.s. if $p < 1/2$ (it is thus convergent a.s. only if $p = 1/2$).

b. Now suppose instead that, for the jth term, we choose '+' with probability $1/2 + \epsilon_j$ and '-' with probability $1/2 - \epsilon_j$, where for each j, $-1/2 \leq \epsilon \leq 1/2$. Find a necessary and sufficient condition on the sequence $\{\epsilon_j\}$ such that $\sum_{j=1}^{\infty}(X_j/j)$ converges a.s.

2. Consider a pure birth process where lifespans are i.i.d. exponentials with mean $1/\lambda$ where λ is unknown. Let S_n be the time of the nth birth event ($S_0 \equiv 0$). Suggest a strongly consistent estimator $\widehat{\lambda}_n$ of λ (i.e. an estimator such that $\widehat{\lambda}_n \to \lambda$ a.s.

3. Let $X_1, X_2, \ldots$ be i.i.d. uniform on (0,1). Consider the arithmetic, geometric and harmonic means respectively, defined by

$$A_n = \frac{1}{n}\sum_{k=1}^{n} X_k$$

$$G_n = (X_1 X_2 \ldots X_n)^{1/n}$$

$$H_n = \frac{n}{\sum_{k=1}^{n} \frac{1}{X_k}}.$$

Find the a.s. limits of these as $n \to \infty$.

4. Let $X_1, X_2, \ldots$ be i.i.d. with mean 0 and finite variance and let $S_n = \sum_{k=1}^{n} X_k$. Show that $S_n/n^{\alpha} \to 0$ a.s. for any $\alpha > 1/2$.