Solutions to Assignment 3, STAT 582

1a. We have $E[X_j/j] = (2p-1)/j$ and $Var[X_j/j] = 4p(1-p)/j^2$. Since $\sum_j Var(X_j/j)$ converges, Kolmogorov's convergence criterion tells us that the behavior of $\sum_j X_j/j$ is the same as that of $\sum_j E[X_j/j] = (2p-1)\sum_j 1/j$ which is ∞ if $p > 1/2$ and $-\infty$ if $p < 1/2$.

b. Here, $Var[X_j/j] = 1 - 4\epsilon_j^2 \leq 1/j^2$ so the sum of the variances is still convergent. Hence $\sum_j X_j/j$ converges if and only if $\sum_j E[X_j/j] = 2\sum_j(\epsilon_j/j$ converges. It is intuitively clear that the probabilites of + or − must converge to 1/2, but note that the condition is stronger: it means that they must converge to 1/2 fast enough for the sum above to converge.

2. $S_n = \sum_{j=1}^n X_j$ where X_j has an exponential distribution with mean $1/(\lambda j)$. Use Corollary 7.4.1 with $b_n = \log n$ to obtain $\sum_j Var[X_j]/(\log j)^2 \leq \sum_j 1/(\lambda j)^2 < \infty$ and hence $(S_n - E[S_n])/\log n \to 0$ a.s. Finally note that

$$\frac{E[S_n]}{\log n} = \frac{\sum_{j=1}^n \frac{1}{j}}{\lambda \log n} \to \frac{1}{\lambda}$$

as $n \to \infty$, i.e.

$$\frac{S_n}{\log n} \to \frac{1}{\lambda} \text{ a.}s.$$

as $n \to \infty$. A strongly consistent estimator of λ is thus $\log n/S_n$.

3. Since $E[X_1] = 1/2$, $A_n \to 1/2$ a.s. by the i.i.d. case of SLLN (Theorem 7.5.1).

Since $\log G_n = \frac{1}{n}\sum_{j=1}^n \log X_j$, Theorem 7.5.1 applies to give $\log G_n \to E[\log X_1]$ a.s. Since $E[\log X_1] = -1$ and $g(x) = e^x$ is continuous, Corollary 6.3.1 (i) applies to give $G_n \to e^{-1}$ a.s.

Let $Y_n = 1/X_n$ and $A_n = 1/H_n$. Then $A_n = \frac{1}{n}\sum_{k=1}^n Y_k$ where the Y_k are i.i.d. with mean $E[Y_1] = \int_0^1 \frac{dx}{x} = \infty$ i.e. $A_n \to \infty$ a.s. which gives $H_n \to 0$ a.s.

Note that the limits in the three cases are different. In fact it holds generally that $A_n \geq G_n \geq H_n$ for any n.

4. Let $\sigma^2 = Var[X_k]$ and use Corollary 7.4.1 with $b_n = n^{-\alpha}$. Then

$$\sum_k \frac{Var[X_k]}{b_k^2} = \sigma^2 \sum_k \frac{1}{n^{2\alpha}}$$

which is finite if and only if $\alpha > 1/2$. Since $E[S_n] = 0$, the sum $\sum_n E[S_n]/n^\alpha$ is always finite and hence $S_n/n^\alpha \to 0$ if $\alpha > 1/2$. If $\alpha \leq 1/2$ the corollary does not apply. However, by Theorem 7.2.1, S_n/n^α does not converge in probability to 0 if $\alpha \leq 1/2$ (condition (ii) fails) and can thus not converge to 0 almost surely. (Also recall that for $\alpha = 1/2$, CLT states that $S_n/\sqrt{n}$ converges in distribution to a normal distribution.)