Assignment 4, STAT 582, due March 24

1. Let $(\Omega, \mathcal{B})$ be a measurable space and let μ and ν be measures. Are the following statements true or false? Give proofs or counterexamples.

a. If $\mu \ll \nu$, then $\mu(B) \leq \nu(B)$ for all $B \in \mathcal{B}$.

b. If $\mu \ll \nu$ and $\nu \ll \mu$, then $\mu(B) = \nu(B)$ for all $B \in \mathcal{B}$.

c. If $\nu \ll \mu$ and μ is not σ-finite, then $\frac{d\nu}{d\mu}$ does not exist.

d. $\frac{d\nu}{d\nu}$ always exists and equals 1.

e. If $\nu \ll \mu$ and μ is σ-finite, then ν is σ-finite.

2. Let μ be counting measure on (R, Borel).

a. Let λ denote Lebesgue measure. Show that $\lambda \ll \mu$ but that $d\lambda/d\mu$ does not exist. Why does this not contradict the Radon-Nikodym theorem?

b.Describe the class of measures on (R, Borel) that are absolutely continuous with respect to μ.

c. Is $\mu \ll \lambda$?

3. Let $(\Omega, \mathcal{B}, P)$ be a probability space and let $\mathcal{G} \subseteq \mathcal{B}$ be a sub-σ-field of $\mathcal{B}$. Let $A, A_1, A_2, ...$ be disjoint sets in $\mathcal{B}$. Show that

a. $P(A^c|\mathcal{G}) = 1 - P(A|\mathcal{G})$

b. $P(\bigcup_{n=1}^{\infty} A_n|\mathcal{G}) = \sum_{n=1}^{\infty} P(A_n|\mathcal{G})$.

4. Let $\Omega = \{-1, 1\}$, $\mathcal{B} = \{\emptyset, \{-1\}, \{1\}, \Omega)$ and P such that $P(\{-1\}) =$

$P(\{1\}) = 1/2$. Further, let $X(\omega) = \omega, Y(\omega) = 2\omega,\ g(x) = x^2$ and consider $\mathcal{G}_1 = \sigma(Y)$ and $\mathcal{G}_2 = \sigma(g(Y))$.

a. What are $\mathcal{G}_1$ and $\mathcal{G}_2$?

b. Find $E[X|\mathcal{G}_1]$, $E[X|\mathcal{G}_2]$, $E[E[X|\mathcal{G}_1]|\mathcal{G}_2]$ and $E[E[X|\mathcal{G}_2]|\mathcal{G}_1]$. Verify the "smallest σ-field always wins" property.

5 Let $X \sim$ unif $[-1, 2]$. Show that $E[X|X^2] = X \cdot I_{\{X>1\}}$.