Solutions to Assignment 4, STAT 582

1a. False. Let \( \nu \) be any measure and \( \mu = 2\nu \). Then \( \mu \ll \nu \) but \( \mu(B) > \nu(B) \) for all sets with \( \nu(B) > 0 \).

b. False. Same example as above.

c. False. Suppose \( \mu \) is not \( \sigma \)-finite and let \( \nu = 2\mu \). Then \( \nu(B) = 2\mu(B) = \int_B 2d\mu \) so \( \frac{d\nu}{d\mu} \equiv 2 \). The existence of a R-N derivative implies absolute continuity but not necessarily \( \sigma \)-finiteness.

d. True in the sense that it always exists and may be taken as \( \equiv 1 \). This follows since \( \nu(B) = \int_B 1 \cdot d\nu \) and if \( \nu \) is \( \sigma \)-finite, the R-N Theorem assures uniqueness. If \( \nu \) is not \( \sigma \)-finite however, other \( f \) may work as well. For example, if \( \nu \) is Lebesgue measure on \((R, \{\emptyset, R\})\), then any \( f \geq 0 \) with \( \int_R f(x)dx = \infty \) will do since \( \nu(\emptyset) = 0 \) and \( \nu(R) = \infty \).

e. False. Let \( \Omega \) be any space and consider the trivial \( \sigma \)-field \( \{\emptyset, \Omega\} \). If \( \mu \) is any measure with \( 0 < \mu(\Omega) < \infty \) and \( \nu(\Omega) = \infty \), then \( \nu \ll \mu \), \( \mu \) is \( \sigma \)-finite but \( \nu \) is not.

2a. If \( \mu(A) = 0 \), then \( A = \emptyset \) and hence \( \lambda(A) = 0 \) also. Hence \( \lambda \ll \mu \). Now suppose there is a function \( f \) such that \( \lambda(A) = \int_A f d\mu \) for all Borel sets \( A \). In particular this holds for the singleton sets, \( \{x\} \) and since \( \lambda(\{x\}) = 0 \) and \( \mu(\{x\}) = 1 \) we must have \( f(x) = 0 \) for all \( x \). But then we get \( 1 = \lambda([0,1]) = \int_0^1 0d\mu = 0 \), a contradiction. Hence no such \( f \) exists, i.e. \( d\lambda/d\mu \) does not exist. Since \( \mu \) is not \( \sigma \)-finite, this does not contradict Radon-Nikodym.

b. Since \( \mu(A) = 0 \) if and only if \( A = \emptyset \) and all measures are 0 on the empty set, all measures on \((R, \text{Borel})\) are absolutely continuous with respect to counting measure.

c. No, for example \( \lambda(\{0\}) = 0 \) but \( \mu(\{0\}) = 1 \).

3a. Try RHS as a candidate for \( P(A^c|G) \). Since \( P(A|G) \) is measurable, so is
$1 - P(A|\mathcal{G})$ and (a) is satisfied. For (b), take $G \in \mathcal{G}$ and note that
\[
\int_G (1 - P(A|\mathcal{G}))dP = \int_G dP - \int_G P(A|\mathcal{G})dP
\]
\[
= P(G) - P(A \cap G)
\]
\[
= P(G \cap A^c)
\]
and hence $P(A^c|\mathcal{G}) = 1 - P(A|\mathcal{G})$.

b. Try RHS as a candidate for $P(\cup_n A_n|\mathcal{G})$. Here (a) is clear since a sum of measurable functions is measurable, but an argument involving increasing limits must be used since the sum is infinite. For (b), take a set $G \in \mathcal{G}$ and note that
\[
\int_G \sum_n P(A_n|\mathcal{G})dP = \sum_n \int_G P(A_n|\mathcal{G})dP = \sum_n P(A_n \cap G) = P((\cup_n A_n) \cap G) = \int_G P(\cup_n A_n|\mathcal{G})dP.
\]

4a. Since $Y^{-1}(\{-2\}) = \{-1\}$ and $Y^{-1}(\{2\}) = \{1\}$, $\mathcal{G}_1 = \sigma(Y) = B$ and since $g(Y) \equiv 4$, $\mathcal{G}_2 = \sigma(g(Y)) = \{\emptyset, \Omega\}$.

b. Since $X$ is measurable w.r.t $\mathcal{G}_1$, $E[X|\mathcal{G}_1] = X$ and since $\mathcal{G}_2$ is trivial, $E[X|\mathcal{G}_2] = E[X] = 0$. Hence $E[E[X|\mathcal{G}_2]|\mathcal{G}_1] = E[0|\mathcal{G}_1] = 0$ and $E[E[X|\mathcal{G}_1]|\mathcal{G}_2] = E[X|\mathcal{G}_2] = 0$ i.e. both equal $E[X|\mathcal{G}_2]$ and since $\mathcal{G}_2 \subseteq \mathcal{G}_1$ this verifies the "smallest wins" property.

5. Let $Z = X \cdot I_{\{X > 1\}}$. Note that on the range $[-1, 2]$, $Z$ is a function of $X^2$: if $X^2 \leq 1$, $Z$ equals 0 and if $X^2 > 1$ $Z$ equals $\sqrt{X^2}$. Hence, $Z = \sqrt{X^2}I_{\{X^2 > 1\}}$ and $Z$ is measurable with respect to $\sigma(X^2)$. Note that $X$ is not a function of $X^2$ on $[-1, 2]$.

Next, take a set $G = \{X^2 \leq a\}$. If $a \leq 1$, we get
\[
\int_G XdP = \frac{1}{3} \int_{-\sqrt{a}}^{\sqrt{a}} xdx = 0
\]
and since $ZI_G = 0$,
\[
\int_G ZdP = 0.
\]

If \( a > 1 \),

\[
\int_G XdP = \frac{1}{3} \int_{-1}^\sqrt{a} x\,dx = \frac{1}{3} \int_1^{\sqrt{a}} x\,dx
\]

and since \( ZI_G = X I_{(1<X^2<a)} = X I_{(1<X<\sqrt{a})} \)

\[
\int_G ZdP = \frac{1}{3} \int_1^{\sqrt{a}} x\,dx
\]

and hence \( E[X|X^2] = X \cdot I_{\{X^2>1\}} \).