Solutions to Assignment 4, STAT 582

1a. False. Let ν be any measure and $\mu = 2\nu$. Then $\mu \ll \nu$ but $\mu(B) > \nu(B)$ for all sets with $\nu(B) > 0$.

b. False. Same example as above.

c. False. Suppose μ is not σ-finite and let $\nu = 2\mu$. Then $\nu(B) = 2\mu(B) = \int_B 2d\mu$ so $\frac{d\nu}{d\mu} \equiv 2$. The existence of a R-N derivative implies absolute continuity but not necessarily σ-finiteness.

d. True in the sense that it always exists and may be taken as $\equiv 1$. This follows since $\nu(B) = \int_B 1 \cdot d\nu$ and if ν is σ-finite, the R-N Theorem assures uniqueness. If ν is not σ-finite however, other f may work as well. For example, if ν is Lebesgue measure on $(R, \{\emptyset, R\})$, then any $f \geq 0$ with $\int_R f(x)dx = \infty$ will do since $\nu(\emptyset)) = 0$ and $\nu(R) = \infty$.

e. False. Let Ω be any space and consider the trivial σ-field $\{\emptyset, \Omega\}$. If μ is any measure with $0 < \mu(\Omega) < \infty$ and $\nu(\Omega) = \infty$, then $\nu \ll \mu$, μ is σ-finite but ν is not.

2a. If $\mu(A) = 0$, then $A = \emptyset$ and hence $\lambda(A) = 0$ also. Hence $\lambda \ll \mu$. Now suppose there is a function f such that $\lambda(A) = \int_A f d\mu$ for all Borel sets A. In particular this holds for the singleton sets, $\{x\}$ and since $\lambda(\{x\}) = 0$ and $\mu(\{x\}) = 1$ we must have $f(x) = 0$ for all x. But then we get $1 = \lambda([0,1]) = \int_0^1 0 d\mu = 0$, a contradiction. Hence no such f exists, i.e. $d\lambda/d\mu$ does not exist. Since μ is not σ-finite, this does not contradict Radon-Nikodym.

b. Since $\mu(A) = 0$ if and only if $A = \emptyset$ and all measures are 0 on the empty set, all measures on (R, Borel) are absolutely continuous with respect to counting measure.

c. No, for example $\lambda(\{0\}) = 0$ but $\mu(\{0\}) = 1$.

3a. Try RHS as a candidate for $P(A^c|\mathcal{G})$. Since $P(A|\mathcal{G})$ is measurable, so is

$1 - P(A|\mathcal{G})$ and (a) is satisfied. For (b), take $G \in \mathcal{G}$ and note that

$$\begin{aligned} \int_G (1 - P(A|\mathcal{G}))dP &= \int_G dP - \int_G P(A|\mathcal{G})dP \\ &= P(G) - P(A \cap G) \\ &= P(G \cap A^c) \end{aligned}$$

and hence $P(A^c|\mathcal{G}) = 1 - P(A|\mathcal{G})$.

b. Try RHS as a candidate for $P(\cup_n A_n|\mathcal{G})$. Here (a) is clear since a sum of measurable functions is measurable, but an argument involving increasing limits must be used since the sum is infinite. For (b), take a set $G \in \mathcal{G}$ and note that

$$\int_G \sum_n P(A_n|\mathcal{G})dP = \sum_n \int_G P(A_n|\mathcal{G})dP = \sum_n P(A_n \cap G) =$$

$$= P((\cup_n A_n) \cap G) = \int_G P(\cup_n A_n|\mathcal{G})dP.$$

4a. Since $Y^{-1}(\{-2\}) = \{-1\}$ and $Y^{-1}(\{2\}) = \{1\}$, $\mathcal{G}_1 = \sigma(Y) = \mathcal{B}$ and since $g(Y) \equiv 4$, $\mathcal{G}_2 = \sigma(g(Y)) = \{\emptyset, \Omega\}$.

b. Since X is measurable w.r.t $\mathcal{G}_1$, $E[X|\mathcal{G}_1] = X$ and since $\mathcal{G}_2$ is trivial, $E[X|\mathcal{G}_2] = E[X] = 0$. Hence $E[E[X|\mathcal{G}_2]|\mathcal{G}_1] = E[0|\mathcal{G}_1] = 0$ and $E[E[X|\mathcal{G}_1]|\mathcal{G}_2] = E[X|\mathcal{G}_2] = 0$ i.e. both equal $E[X|\mathcal{G}_2]$ and since $\mathcal{G}_2 \subseteq \mathcal{G}_1$ this verifies the "smallest wins" property.

5. Let $Z = X \cdot I_{\{X>1\}}$. Note that on the range $[-1, 2]$, Z is a function of X^2: if $X^2 \leq 1$, Z equals 0 and if $X^2 > 1$ Z equals $\sqrt{X^2}$. Hence, $Z = \sqrt{X^2} I_{\{X^2>1\}}$ and Z is measurable with respect to $\sigma(X^2)$. Note that X is not a function of X^2 on $[-1, 2]$.

Next, take a set $G = \{X^2 \leq a\}$. If $a \leq 1$, we get

$$\int_G X dP = \frac{1}{3} \int_{-\sqrt{a}}^{\sqrt{a}} x dx = 0$$

and since $ZI_G = 0$,

$$= \int_G ZdP = 0.$$

If $a > 1$,

$$\int_G XdP = \frac{1}{3}\int_{-1}^{\sqrt{a}} xdx = \frac{1}{3}\int_{1}^{\sqrt{a}} xdx$$

and since $ZI_G = XI_{\{1<X^2<a\}} = XI_{\{1<X<\sqrt{a}\}}$

$$\int_G ZdP = \frac{1}{3}\int_{1}^{\sqrt{a}} xdx$$

and hence $E[X|X^2] = X \cdot I_{\{X^2>1\}}$.