Assignment 5, STAT 582, due April 21

1. Let $X_1, X_2, ...$ be i.i.d. random variables such that

$$X_n = \begin{cases} 0 & \text{with probability } 1/2 \\ 2 & \text{with probability } 1/2, \end{cases}$$

let $Y_n = X_1 \cdot X_2 \cdot ... \cdot X_n$ and $\mathcal{B}_n = \sigma(X_1, ..., X_n)$.

a. Show that Y_n is a martingale with respect to $\mathcal{B}_n$.

b. Does Y_n converge almost surely? If so, to what? Does Y_n converge in L_1?

2. Let S_n be a symmetric simple random walk starting in 0 and let $\nu = \inf\{n : S_n = -a \text{ or } S_n = a\}$ (symmetric absorbing barriers). Let $Y_n = S_n^4 - 6nS_n^2 + 3n^2 + 2n$.

a. Show that Y_n is a martingale (with respect to $\mathcal{B}_n = \sigma(X_0, ..., X_n)$).

b. It can be shown that the Optional Stopping Theorem applies to Y_n. Use this to compute $Var[\nu]$.

3. Let ν and τ be stopping times. Which of the following are stopping times?

a. $\min(\nu, \tau)$ **b.** $\max(\nu, \tau)$ **c.** $\nu + \tau$ **d.** $\nu + 1$ **e.** $\nu - 1$

4. Let $\{Z_n\}$ be a branching process and let q be the extinction probability. Let $Y_n = q^{Z_n}$ and let $\mathcal{B}_n = \sigma(Z_0, ..., Z_n)$. Show that Y_n is a martingale w.r.t. $\mathcal{B}_n$.

b. Does Y_n converge almost surely? To what? Does Y_n converge in L_1?