Solutions to Assignment 6, STAT 582

1a. Since Y_n is a function of $X_1, ..., X_n$ it is measurable with respect to $\mathcal{B}_n$. Further,

$$E[Y_{n+1}|\mathcal{B}_n] = E[Y_n X_{n+1}|\mathcal{B}_n] = Y_n E[X_{n+1}] = Y_n$$

since Y_n is measurable with respect to $\mathcal{B}_n$, X_{n+1} is independent of $\mathcal{B}_n$ and $E[X_{n+1}] = 1$.

b. Since the event $\{X_n = 0 \text{ for some } n\}$ has probability one, $Y_n \to 0$ a.s. Since $E[|Y_n - 0|] = E[Y_n] = 1$, Y_n does not converge in L_1.

2a. Use the facts that $S_{n+1} = S_n + X_{n+1}$ and that the X_k are i.i.d. with mean 0 to show that

$$E[S_{n+1}^4|\mathcal{B}_n] = S_n^4 + 6S_n^2 + 1$$

and

$$E[S_{n+1}^2|\mathcal{B}_n] = S_n^2 + 1.$$

Hence,

$$E[Y_{n+1}|\mathcal{B}_n] = S_n^4 + 6S_n^2 + 1 - 6(n+1)(S_n^2 + 1) + 3(n+1)^2 + 2(n+1)$$

$$= S_n^4 - 6nS_n^2 + 3n^2 + 2n = Y_n.$$

b. By OST, $E[Y_\nu] = E[Y_0] = 0$ i.e.

$$0 = E[S_\nu^4] - 6E[\nu S_\nu^2] + 3E[\nu^2] + 2E[\nu] = a^4 - 6a^2a^2 + 3E[\nu^2] + 2a^2$$

where we have used the facts that $E[\nu] = a^2, S_\nu^2 \equiv a^2, S_\nu^4 \equiv a^4$. This gives

$$E[\nu^2] = \frac{1}{3}(5a^4 - 2a^2)$$

which gives

$$Var[\nu] = \frac{1}{3}(5a^4 - 2a^2) - a^4 = \frac{2}{3}(a^4 - a^2).$$

3. All except **e** are stopping times. First note that $\mathcal{B}_n$ is an increasing sequence and therefore $\{\nu = k\} \in \mathcal{B}_k \subset \mathcal{B}_n, k = 1, ..., n$ (similarily for τ).

a. $\{\min(\sigma, \tau) = n\} = (\{\sigma = n\} \cap \{\tau \geq n\}) \cup (\{\sigma \geq n\} \cap \{\tau = n\}) \in \mathcal{B}_n$

b. As in **a.** with a $\cup$ instead of $\cap$ and vice versa.

c. $\{\nu + \tau = n\} = \sum_{k=0}^{n} \{\nu = k\} \cap \{\tau = n - k\} \in \mathcal{B}_n$.

d. $\{\tau + 1 = n\} = \{\tau = n - 1\} \in \mathcal{B}_n$.

e. $\{\tau - 1 = n\} = \{\tau = n + 1\}$, not necessarily in $\mathcal{B}_n$.

4a. Since $Z_{n+1} = \sum_{k=1}^{Z_n} X_k$ where the X_k are i.i.d we get

$$E[Y_{n+1}|\mathcal{B}_n] = E[q^{Z_{n+1}}|Z_n] = \prod_{k=1}^{Z_n} E[q^{X_k}] = q^{Z_n} = Y_n$$

where we use the fact that $E[q^{X_k}] = \varphi(q) = q$.

b. Yes. Recall that $Z_n \to Z$ a.s. where $Z = 0$ w.p. q and $Z = \infty$ w.p. $1 - q$. Hence $Y_n \to Y$ a.s where $Y = 1$ w.p. q and $Y = 0$ w.p. $1 - q$. Note that the limit here is a random variable, not a constant unless $q = 1$, in which case $Y_n \equiv 1$ and $q = 0$, in which case $Y_n \equiv 0$.

Since $|Y_n - Y| \to 0$ and $|Y_n - Y| \leq |Y_n| + |Y| \leq 2$, DCT applies to give $E[|Y_n - Y|] \to 0$ i.e. $Y_n \xrightarrow{L_1} Y$.

The following argument that Y_n does not converge in L_1 is **INCORRECT**: Since $Y = 1$ with probability q and 0 with probability $1 - q$, we get

$$E[|Y_n - Y|] = E[|Y_n - 1|]q + E[|Y_n - 0|](1-q) =$$

$$= (1 - E[Y_n])q + E[Y_n](1-q) = 2q(1-q) \not\to 0.$$

The problem is that Y and Y_n are **DEPENDENT** so that when conditioning on Y, the mean $E[|Y_n - Y|]$ changes. Thus, $E[|Y_n - Y||Y = 0]$ is **NOT** equal to $E[|Y_n|] = q$; conditioning on $Y = 0$ means that we condition on $Z_n \to \infty$ which changes the distribution of Z_n (it can for example certainly not be 0 anymore) and hence that of Y_n. Think about this!