Final Exam, STAT 582.

1. Suppose we have a.s. finite random variables such that $X_n \to X$ and $Y_n \to Y$. Show that $X_n Y_n \to XY$ if all the "$\to$" denote:

(a) convergence almost surely

(b) convergence in probability.

Give examples to show that this is not necessarily true if "$\to$" denotes:

(c) convergence in L_1

(d) convergence in distribution.

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2. Consider a sequence of urns, $U_1, U_2, ...$ such that urn number k contains 1 red and $k-1$ white balls. Pick one ball at random from each urn, starting with U_1 and proceeding. Let S_n be the total number of red balls after you have picked from the nth urn. Show that

$$\frac{S_n}{\log n} \to 1 \quad a.s.$$

You may use without proof the fact that

$$\sum_{k=2}^{\infty} \frac{1}{k(\log k)^2} < \infty.$$

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3. Let X and Y be i.i.d. random variables with expectation μ. We a re looking for $E[X|X+Y]$. Intuitively, given the value of $X+Y$, X should on the average contribute with one half and Y with the other, i.e. $E[X|X+Y] = (X+Y)/2$. But one could also argue that Y is on the average μ and given that $X+Y=z$, we should expect X to be $z-\mu$ so that $E[X|X+Y] = X+Y-\mu$. Which is correct and why?

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4. Let $X_1, X_2, ...$ be i.i.d random variables such that $X_k = 1$ with probability p (where $0 < p < 1$) and -1 with probability $q = 1-p$. Let $S_n = \sum_{k=1}^n X_k$ and $Y_n = (q/p)^{S_n}$.

(a) Show that Y_n is a martingale with respect to $\mathcal{F}_n = \sigma(X_1, ..., X_n)$.

(b) For which values of p does Y_n converge almost surely? In L_1? To what?

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5. Let $X_1, X_2, ...$ be i.i.d such that

$$X_k = \begin{cases} -1 & \text{with probability } 1-p \\ \\ 1 & \text{with probability } p \end{cases}$$

where $p > 1/2$, and let $S_n = \sum_{k=1}^n X_k$ (and $S_0 \equiv 0$). This is called an *asymmetric random walk*.

(a) Show that $S_n \to \infty$ a.s.

(b) Let $Y_n = S_n - n(2p-1)$. Show that Y_n is a martingale with respect to $\mathcal{B}_n = \sigma(X_1, ..., X_n)$.

(c) Let ν be the first time the walk hits 1. Give a strict definition of ν and show that it is a stopping time.

(d) It can be shown that the Optional Stopping Theorem applies (you do not have to show this). Use this to find $E[\nu]$.

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6. Let $X_1, X_2, ...$ be i.i.d. with probability density function

$$f(x) = 3x^2, \quad 0 \le x \le 1$$

and define the harmonic mean as

$$H_n = \frac{n}{\sum_{k=1}^{n} \frac{1}{X_k}}.$$

(a) Find the a.s. limit of H_n as $n \to \infty$.

(b) The delta method says that if $Y_1, Y_2, ...$ are i.i.d. with mean μ and variance σ^2, then the function $g(\bar{Y})$ is approximately normal with mean $g(\mu)$ and variance $\sigma^2 g'(\mu)^2/n$, provided $g'(\mu) \neq 0$. Use this to find the approximate distribution of H_n.