Solutions to Final Exam, STAT 582

1(a) Let A be the set where $X_n \to X$ and B the set where $Y_n \to Y$ (i.e. $A = \{\omega : X_n(\omega) \to X(\omega)\}$ etc). Then $A \cap B \subseteq \{X_n Y_n \to XY\}$ (if $X_n(\omega) \to X(\omega)$ and $Y_n(\omega) \to X(\omega)$, then $X_n(\omega)Y_n(\omega) \to X(\omega)Y(\omega)$ but the latter convergence could still hold even if not both the former do). Since $P(A) = 1$ and $P(B) = 1$, we get $P(A \cap B) = 1 - P(A^c \cup B^c) \geq 1 - (P(A^c) + P(B^c) = 1$ i.e. $P(A \cap B) = 1$ i.e. $X_n Y_n \to XY$ a.s.

(b) Let $\{n_k\}$ be an arbitrary subsequence and take a further subsequence $\{n_k^{'}\} \subseteq \{n_k\}$ such that $X_{n_k^{'}} \to X$ a.s. Now take yet a further subsequence $\{n_k^{''}\} \subseteq \{n_k^{'}\}$ such that $Y_{n_k^{''}} \to Y$ a.s. Thus, by (i) ablve, $X_{n_k^{''}} Y_{n_k^{''}} \to XY$ a.s. which proves that $X_n Y_n \xrightarrow{P} XY$.

(c) Take a sequence $X_n \geq 0$ such that X_n converges to 0 in L^1 but not in L^2 (for example $X_n = n$ with probability $1/n^2$ and 0 with probability $1 - 1/n^2$). Let $Y_n \equiv X_n$ to obtain $X_n \xrightarrow{L^1} 0$, $Y_n \xrightarrow{L^1} 0$ but $X_n Y_n = X_n^2 \stackrel{L^1}{\not\to} 0$.

(d) Let X_n be -1 or 1 with equal probabilities and let $Y_n = 1/X_n$ Then X_n and Y_n have the same distributions as X_1 and $-X_1$ so we have $X_n \xrightarrow{d} X_1$ and $Y_n \xrightarrow{d} -X_1$. But $X_n Y_n \equiv 1 \stackrel{d}{\not\to} XY \equiv -1$.

2. With $A_k = \{$ ball from kth urn is red$\}$, we get $S_n = \sum_{k=1}^{n} I_{A_k}$ where the terms are independent and the kth term has mean $1/k$ and variance $1/k(1 - 1/k)$. Now apply Corollary 7.4.1.

3. $E[X|X+Y] = (X+Y)/2$ is correct. You can either use the definition, or note that

$$\begin{aligned} X + Y &= E[X+Y|X+Y] \\ &= E[X|X+Y] + E[Y|X+Y] \\ &= 2E[X|X+Y] \end{aligned}$$

since X and Y are i.i.d.

4 (a) Since Y_n is a function of $X_1, ..., X_n$, it is measurable with respect to $\mathcal{F}_n$. Further, since S_n is measurable with respect to and X_{n+1} independent of $\mathcal{F}_n$ we get

$$E[Y_{n+1}|\mathcal{F}_n] = E\left[\left(\frac{q}{p}\right)^{S_{n+1}} |\mathcal{F}_n\right] = E\left[\left(\frac{q}{p}\right)^{S_n} \left(\frac{q}{p}\right)^{X_{n+1}} |\mathcal{F}_n\right] =$$

$$= \left(\frac{q}{p}\right)^{S_n} E\left[\left(\frac{q}{p}\right)^{X_{n+1}}\right] = Y_n(\frac{q}{p}p + (\frac{q}{p})^{-1}q = Y_n(q+p) = Y_n.$$

(b) Since Y_n is a nonnegative martingale, it converges almost surely. To find the limit, first note that if $p = q = 1/2$, we have $Y_n \equiv 1$ so in this case trivially $Y_n \to 1$ a.s. and in L_1. If $p > q$, SLLN gives that $S_n/n \to 2p - 1$ a.s. and since $2p - 1 > 0$, this means that $S_n \to \infty$ a.s. Since $q/p < 1$, we get $Y_n \to 0$ a.s. A similar argument shows that $Y_n \to 0$ a.s. also if $p < q$. Since $E[Y_n] \equiv 1$, there cannot be convergence in L_1 in any of these cases.

5 (a) By SLLN, $S_n/n \to \mu$ a.s. where $\mu = E[X] = 2p - 10$ and since this is positive and $n \to \infty$, we must have $S_n \to \infty$ a.s.

(b) Clearly Y_n is measurable with respect to $\mathcal{B}_n$. Further,

$$\begin{aligned}
E[Y_{n+1}|\mathcal{B}_n] &= E[S_{n+1}|\mathcal{B}_n] - (n+1)(2p-1) \\
&= E[S_n|\mathcal{B}_n] + E[X_{n+1}|\mathcal{B}_n] - (n+1)(2p-1) \\
&= S_n + 2p - 1 - (n+1)(2p-1) \\
&= S_n - n(2p-1) \\
&= Y_n
\end{aligned}$$

and hence Y_n is a martingale.

(c) $\nu = \min\{n : S_1 < 1, ..., S_{n-1} < 1, S_n = 1\}$. Since

$$\{\nu = n\} = \{S_1 < 1, ..., S_{n-1} < 1, S_n = 1\} \in \mathcal{B}_n$$

since each S_k is a function of $X_1, ..., X_k$. Hence ν is a stopping time.

(d) The OST gives that $E[Y_\nu] = E[Y_0]$. Since $E[Y_0] = 0$ and $S_\nu \equiv 1$ we get

$$\begin{aligned} E[Y_\nu] &= E[S_\nu] - E[\nu](2p-1) \\ &= 1 - E[\nu](2p-1) \\ &= 0 \end{aligned}$$

which gives

$$E[\nu] = \frac{1}{2p-1}.$$

6(a) Let $Y_k = 1/X_k$. Then

$$E[Y_k] = \int_0^1 3x dx = 3/2$$

and hence

$$\bar{Y} \to 3/2 \ \ a.s.$$

Since $H_n = 1/\bar{Y}$, a continuous function of $\bar{Y}$, we get

$$H_n \to 2/3 \ \ a.s.$$

(b) The delta method applies to the Y_k and $\bar{Y}$. The function is $g(x) = 1/x$ which has $g'(x) = -1/x^2$. The variance of Y_k is

$$\mathrm{Var}[Y_k] = \int_0^1 3dx - (3/2)^2 = 3/4$$

which gives that, approximately,

$$H_n \sim N\left(\frac{2}{3}, \frac{4}{27n}\right).$$