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Short communication

On the time for Brownian motion to visit every point on a circle

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ABSTRACT

Consider a Wiener process W on a circle of circumference L. We prove the rather surprising result that the Laplace transform of the distribution of the first time, θ_L, when the Wiener process has visited every point of the circle can be solved in closed form using a continuous recurrence approach.

1. Introduction

Consider a Wiener process on a circle of circumference L. The distribution of the first time, θ_L, when the Wiener process has visited every point of the circle is equivalent, via the natural bijection between and interval of the form $[b, b+L)$ on the real line and a circle of circumference L, to the distribution of the first time when the range of the Wiener process on the real line is of length L. This distribution is well-known and it has the following Laplace transform: [see, for example, (Borodin and Salminen, 2002), p. 242]

$$\mathbb{E}\left[e^{-s\theta_L}\right] = \frac{1}{\cosh^2\left(L\sqrt{\frac{s}{2}}\right)}, \quad s \geq 0. \tag{1}$$

Feller (1951), in writing about the range of a Wiener process, did so using explicit probability density calculations. Imhof (1986) discovered Laplace transform for the first time, θ_L, when the Wiener process has visited every point of the circle, again via explicit probability density calculations. Further computations employing the Laplace transform for θ_L were presented in Vallois (1993). However, in departure from these previous works, we prove the result in Eq. (1) using a continuous recurrence setup. We do so by calculating the left hand side in terms of random variables representing how long it takes to cover a range of length L, *given that one is already at an endpoint of a range of length a* (which counts as being covered already). This is the idea behind the definition of $\theta_{a,L}$, which is defined in Section 2.

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Key to our recurrence will be the concept of a *switchback*. Imagine we pick some $a \in \mathbb{R}^+$ that is less than L. Consider the maximum, M_a, of W until the first visit to the point, $-a$ on the negative half-axis. (Here, $M_a > 0$; otherwise, the process would have moved directly from 0 to $-a$, which occurs w.p. 0) We call the time of this first visit τ_{-a}. We say that a "*switchback*" occurs when W hits $-a$ before the length of the range, $a + M_a$, is L. Formally, let $\mathbb{1}_{a,L}$ be the indicator random variable for the event of a switchback, defined as follows:

$$\mathbb{1}_{a,L} = \begin{cases} 1 & \text{if } \inf\{t : 0 \le t < \infty \mid W_t = -a\} \le \inf\{t : 0 \le t < \infty \mid W_t = L - a\} \\ 0 & \textit{otherwise}. \end{cases}$$

After a switchback, the process continues from $-a$ with a starting range of $M_a + a$ (i.e., the interval $[-a, M_a]$ has been covered). By translation and reflection invariance, as well as the symmetry of Brownian motion, we may just as well assume that we are at the point 0 and have covered the interval $[-(a + M_a), 0]$. We then repeat the process and say that a second switchback occurs if we reach $-(a + M_a)$ before covering a range of length L. To summarize:

Step 1: We start our process at the right hand end of $[-a, 0]$ and we consider this interval as already being covered. M_a is the maximal value attained before the time τ_a that we first hit $-a$. The total range is $a + M_a$. If $M_a \ge L - a$, then we have covered an interval of length L before reaching $-a$, and no switchback occurs. If not, a switchback occurs and we continue to Step 2.

Step 2: We have covered a range of length $a+M_a$. Without loss of generality, we consider the interval $[-(a+M_a), 0]$ to have been covered. Let $-(a+M_a) := -a'$, and start the process on the right hand end of $[-a', 0]$. If $M_{a'} \ge L - M_{a'}$, no switchback occurs. Otherwise, another switchback occurs and we continue to Step 3.

Step 3: We have covered a range of length $a' + M_{a'}$. Without loss of generality, we consider the interval $[-(a' + M_{a'}), 0]$ to have been covered. Let $-(a' + M_{a'})$ be called $-a''$, and start the process on the right hand end of $[-a'', 0]$. If $M_{a''} \ge L - a''$, a switchback occurs. Otherwise, continue Step 3 recursively until a range of length L has been covered.

Steps 1–3 are illustrated in Fig. 1.

In Section 3 we prove that the recurrence can be solved in closed form. In Section 4 we prove that the number $\nu = \nu_{a,L}$, of switchbacks before covering an interval of length L has a Poisson distribution with parameter $\lambda = \log \frac{L}{a}$. Thus, as $a \downarrow 0$, the number of switchbacks goes to infinity at a logarithmic rate.

2. Solving the recurrence

We proceed to solve for the recurrence. First, consider a Wiener process $W(t),\ t \ge 0$. For each fixed, $a > 0$, let M_a denote the maximum positive value of $W(t)$ before the first hitting time of $-a$. Assuming that $L - a$ is positive, we have

$$\mathbb{P}(M_a \le y) = \mathbb{P}\left(\tau_{-a} < \tau_y\right) = \frac{y}{a + y},$$

by the logic of the gambler's ruin.

Let $I(t)$ be the range of the Wiener process up to time t. Define $\theta_{a,L}$ to be the random variable representing the time until $I(t) \cup [-a, 0]$ has length L. We proceed by defining

$$f(s, a, L) := \mathbb{E}\left[\exp\left(-s\theta_{a,L}\right)\right], \tag{2}$$

where $f(s, a, L)$ is considered a function of a with s and L being held constant. By abuse of notation, we label $f(s, a, L)$ as $f(a)$.

Let us define the following functions

$$F(s, y) = \mathbb{E}\left[\exp\left(-s\tau_{-a}\right) \mathbb{1}_{\tau_{-a} < \tau_y}\right] \quad \text{and} \quad G(s, y) = \mathbb{E}\left[\exp\left(-s\tau_y\right) \mathbb{1}_{\tau_y < \tau_{-a}}\right]. \tag{3}$$

We now employ the well-known fact (see Borodin and Salminen, 2002, amongst other sources), that for any c,

$$\exp\left(cW(t) - \frac{c^2}{2}t\right) \quad t \ge 0 \tag{4}$$

is a martingale. If $s = \frac{c^2}{2}$, we easily obtain the following standard and well known forms of $F(s, y)$ and $G(s, y)$ (see Borodin and Salminen, 2002, amongst other sources),

$$F(s, y) = \frac{\sinh cy}{\sinh\left(c(a + y)\right)} \quad \text{and} \quad G(s, y) = \frac{\sinh ca}{\sinh\left(c(a + y)\right)}. \tag{5}$$

Continuing from above, our goal is to write a recurrence for $f(a)$ in terms of $f(a + y)$ for $0 < y \le L - a$. To do so, we define $f(a)$ using indicator functions. With the process starting at 0, let the first indicator function represent the case of a switchback, in which $-a$ is hit before the length of the range is L. Let the second indicator function denote the case of no switchback. We may then write

$$f(a) = \underbrace{\mathbb{E}\left[\exp\left(-s\theta_{a,L}\right) \mathbb{1}_{\tau_{-a} < \tau_{L-a}}\right]}_{\text{switchback}} + \underbrace{\mathbb{E}\left[\exp\left(-s\theta_{a,L}\right) \mathbb{1}_{\tau_{L-a} < \tau_{-a}}\right]}_{\text{no switchback}}. \tag{6}$$

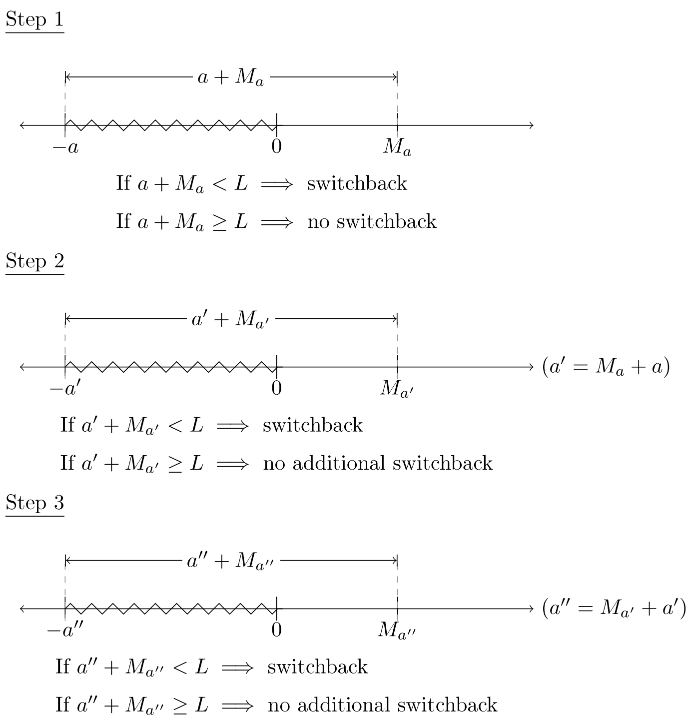

Fig. 1. Illustration of Steps 1–3 in Section 1.

Letting $y = L - a$, and using the expression for $G(s, y)$ in equation (5), we have:

$$G(s, L-a) = \frac{\sinh ca}{\sinh cL}, \tag{7}$$

which is exactly the "no switchback" term. To calculate the switchback term, we integrate over all possible values of M_a, from 0 to $L - a$, using y as a dummy variable. $f(a)$ becomes:

$$f(a) = \frac{\sinh ca}{\sinh cL} + \int_0^{L-a} f(a+y) \frac{d}{dy} F(s, y)\, dy\,, \quad 0 < a \leq L. \tag{8}$$

We note that it is possible for a to be L since the original definition of $f(s, a, L)$ gives $f(L) = 1$.

This recurrence structure is an integral equation. The key idea is that Eq. (8) shows that the expected time it takes to get from point a to point b, conditional on starting at a left most point of an interval, can be found by integrating over all possible left most points of the subsequent path.

Remark 2.1. This continuous recurrence approach presents an enormously valuable alternative to a direct density calculation. This approach should be helpful in many applied statistical settings in which such a calculation is intractable!

3. A closed form solution for the recurrence

Theorem 3.1. *The recurrence structure in Eq.* (8) *can be solved in closed form. Letting* $a \downarrow 0$, *we obtain* $f(0) = \frac{1}{\cosh^2\left(L\sqrt{\frac{s}{2}}\right)}$.

Proof. Using the expression for $F(s, y)$ in equation (5) we have

$$f(a) = \frac{\sinh ca}{\sinh cL} + \int_0^{L-a} f(a+y) \frac{d}{dy} \left[\frac{\sinh cy}{\sinh c(a+y)} \right] dy, \quad 0 < a \leq L. \tag{9}$$

Differentiating, we obtain:

$$f(a) = \frac{\sinh ca}{\sinh cL} + \int_0^{L-a} \frac{c \sinh ca}{\sinh^2 c(a+y)} f(a+y) dy, \quad 0 < a \leq L.$$

Finally, substituting $x = a + y$ gives the integral equation

$$f(a) = \frac{\sinh ca}{\sinh cL} + \int_a^L \frac{c \sinh ca}{\sinh^2 cx} f(x)dx, \quad 0 < a \le L.$$

Fortunately, this is easy to solve: we divide by $\sinh ca$ and let

$$g(x) = \frac{f(x)}{\sinh cx}$$

to arrive at:

$$g(a) = \frac{1}{\sinh cL} + \int_a^L \frac{c}{\sinh cx} g(x)dx, \quad 0 < a < L. \tag{10}$$

Differentiating with respect to a, we obtain the following differential equation for g

$$g'(a) = -\frac{c}{\sinh ca} g(a). \tag{11}$$

Noting that $f(L) = 1$,

$$g(L) = \frac{1}{\sinh cL}. \tag{12}$$

We now have that

$$g(a) = \frac{1}{\sinh cL} \exp\left(\int_a^L \frac{c}{\sinh cu} du\right),$$

which is a unique solution to Eq. (11) with (12) as its initial condition.

Further,

$$f(a) = \frac{\sinh ca}{\sinh cL} \exp\left(\int_a^L \frac{cdu}{\sinh cu} du\right).$$

We now let $a \downarrow 0$. The limit is

$$f(0) = \mathbb{E}\left[\exp\left(-s\theta_L\right)\right] = \lim_{a \to 0} \exp\left(\int_a^L \left(\frac{c}{\sinh cu} - \frac{c \cosh cu}{\sinh cu}\right) du\right). \tag{13}$$

Combining the fractions, integrating, and letting $c = \sqrt{2s}$, we obtain

$$f(0) = \mathbb{E}\left[\exp\left(-s\theta_L\right)\right] = \frac{2}{1 + \cosh cL} = \frac{1}{\cosh^2\left(L\sqrt{\frac{s}{2}}\right)}. \tag{14}$$

Since by Oberhettinger and Badii (1973)

$$\int_0^\infty \exp(-st) \exp\left(-\frac{a^2}{2t}\right) \frac{a}{\sqrt{2\pi}} t^{-\frac{3}{2}} dt = \exp\left(-a\sqrt{2s}\right), \tag{15}$$

we can expand $\left(\cosh^2 \sqrt{\frac{s}{2}}\right)^{-1}$ in powers of $e^{-\sqrt{2s}}$ to obtain an infinite series representation of the density of θ_1. Namely, for $L = 1$, we can write:

$$\begin{aligned}\int_0^\infty e^{-st} p_{\theta_1}(t)dt &= 4\exp\left(-\sqrt{2s}\right)\left(1 + \exp\left(-\sqrt{2s}\right)\right)^{-2} \\ &= \sum_{n=0}^\infty 4(-1)^n (n+1) \exp\left(-(n+1)\sqrt{2s}\right).\end{aligned} \tag{16}$$

Since the Laplace transform is invertible, it suffices to find a formula for $p_{\theta_1}(t)$ that makes the above equation true. If we take

$$p_{\theta_1}(t) = \sum_{n=0}^\infty 4 \frac{(-1)^n (n+1)^2}{\sqrt{2\pi} t^{\frac{3}{2}}} \exp\left(-\frac{(n+1)^2}{2t}\right)$$

and plug this into the left hand side of (16), then using (15) gives us the equality (16). Since the amount of time to cover a range of length L is equal to L^2 multiplied by the amount of time to cover a range of length 1, we write: $\theta_L \sim L^2\theta_1$. Thus, the density of θ_L is

$$p_{\theta_L}(t) = \frac{p_{\theta_1}\left(\frac{t}{L^2}\right)}{L^2}. \quad \square \tag{17}$$

4. The number of switchbacks

The result that the number of switchbacks is distributed as a Poisson random variable comes naturally when one accounts for the Markov and scaling properties of Brownian motion. The formal proof of the result follows.

Theorem 4.1. *The number of switchbacks has a Poisson distribution with parameter* $\lambda = \log\left(\frac{L}{a}\right)$.

Proof. The argument that the number of switchbacks has a Poisson distribution with parameter $\lambda = \log\left(\frac{L}{a}\right)$ is similar to the argument in Section 3 used to obtain the distribution of θ_L. We begin by defining

$$f(a) = f(a, L, t) = \mathbb{E}\left[t^{\nu_{L,a}}\right], \tag{18}$$

where t is a dummy variable and $\nu_{L,a}$ is the number of switchbacks starting from an endpoint of an interval of length $a > 0$ before the interval grows to length L. $\nu_{L,a} = 0$ with probability $\frac{a}{L}$ and

$$\mathbb{P}(M_a \le y) = \frac{y}{a+y}.$$

We rewrite $f(a)$ as we did in Section 3, splitting it up by indicator random variables which account for whether or not a switchback has occurred. If no switchbacks have occurred, $\nu_{L,a} = 0$, and thus:

$$\mathbb{E}\left[t^{\nu_{L,a}}\mathbb{1}_{M_a > L-a}\right] = \mathbb{E}\left[\mathbb{1}_{M_a > L-a}\right] = \frac{a}{L}. \tag{19}$$

Employing Eq. (19) and recalling that a switchback occurs when $M_a < L - a$, we write:

$$f(a) = \frac{a}{L} + \int_0^{L-a} t^{\nu_{L,y+a}}\mathbb{P}(M_a \in dy)\, dy,$$

which simplifies to

$$f(a) = \frac{a}{L} + \int_0^{L-a} tf(a+y)\frac{d}{dy}\left[\frac{y}{a+y}\right] dy.$$

Thus,

$$f(a) = \frac{a}{L} + t\int_0^{L-a} f(a+y)\frac{a}{(a+y)^2}dy = \frac{a}{L} + at\int_0^{L}\frac{f(x)}{x^2}dx. \tag{20}$$

It is straightforward to verify directly, or to use the differential equation argument used above, to find the Laplace transform of θ_L, so that

$$f(a) = \exp\left(\ln\left(\frac{L}{a}\right)(t-1)\right).$$

Since the above is the form of the Laplace transform of the Poisson distribution, we have the desired result that the number of switchbacks is distributed Poisson with mean $\lambda = \ln\left(\frac{L}{a}\right)$. $\square$

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