STAT 552 Homework 1

Due date: In class on Thursday, September 11th, 2003

Instructor: Dr. Rudolf Riedi

1. (a) Let X be the outcome of tossing a fair die. What is the pgf $P_X(s)$ of X?

   (b) Toss a die repeatedly. Let a_n be the number of ways in which it is possible to arrive at exactly n as the sum of the faces of the die. So, $a_1 = 1$ since $n = 1$ can be achieved in only 1 way, i.e., by getting a 1 in the first toss. Also, $a_3 = 4$ since there are four ways to arrive at 3 as the sum: 3, $2+1$, $1+2$, $1+1+1$. Find the generating function of a_n. Hint: recursion.

2. Let X_n $(n \geq 1)$ be iid Bernoulli random variables with

$$P[X_i = 1] = p = 1 - P[X_i = 0] \tag{1}$$

   and let $S_n = X_1 + \ldots X_n$.

   (a) Show that

$$P[S_{n+1} = k] = pP[S_n = k-1] + (1-p)P[S_n = k]. \tag{2}$$

   (b) Multiply equation (2) by s^k to obtain a recursion formula for the probability generating function of S_n and solve it.

   (c) Alternatively to 2b, derive a formula for the pgf of S_n using $P_X(s)$.

   (d) Using the pgf, verify that S_n has a Binomial distribution, i.e., $S_n \sim b(k; n, p)$.

3. Let X_n $(n \geq 1)$ and N be non-negative integer valued random variables, all with finite mean and finite variance. Assume that X_n $(n \geq 1)$ are iid, and independent of N.

   Let $S_n = X_1 + \ldots X_n$. Using pgf, verify that

$$\mathrm{Var(S_N) = \mathbb{E}[N]Var(X_1) + (\mathbb{E}[X_1])^2 Var(N)} \tag{3}$$

4. Let X and Y be jointly distributed non-negative integer valued random variables. For $|s| < 1$ and $|t| < 1$ define

$$P_{X,Y}(s,t) := \sum_{j \geq 0, k \geq 0} s^j t^k P[X = j, Y = k]. \tag{4}$$

   Prove that X and Y are independent if and only if

$$P_{X,Y}(s,t) = P_X(s) P_Y(t) \tag{5}$$

   for $|s| < 1$ and $|t| < 1$. Hint: Use Fubini to convert the right side into one double-indexed sum and compare coefficients.