STAT 552 Homework 6

Recommended due: Wednesday, November 26, 2003 (will be graded by Dec 2)
Extension Dec 1 granted (will be graded by Dec 6)

Instructor: Dr. Rudolf Riedi

25. (Age and excess life)

    Let N_t be a pure renewal process. Let $A(t)$ be the age of the current item, and $B(t)$ its excess life time as usual. Also, set as usual

    $$F_0(x) = \frac{1}{\mu} \int_0^x (1 - F(u))du. \tag{1}$$

    (a) Recall the renewal equation of the tail probability $P[B(t) > x]$. Using Blackwell's theorem show that $P[B(t) \leq x] \to F_0(x)$.

    (b) Similarly, show that $P[A(t) \leq x] \to F_0(x)$.

    Still with the same setting fix now two positive numbers x and y and define

    $$Z(t) := P[A(t) > x, B(t) > y]. \tag{2}$$

    (c) Write a renewal equation for $Z(t)$.

    (d) Compute $\lim_{t\to\infty} Z(t)$.

26. (On-off system)

    Consider a machine of a given type which stay "operative" for an exponential length of time of mean duration $1/\lambda$. When a breakdown occurs, repairs are started immediately and last for an exponential length of time of mean duration $1/\mu$. Repairs return the machine into the original "operative" state. Let Z_t be the state of the machine at time t.

    (a) Explain why we may write Z_t as a continuous time Markov chain with state space $S = \{0, 1\}$, holding time parameters $\lambda(0) = \lambda$ and $\lambda(1) = \mu$ and transition matrix of the underlaying discrete Markov chain as

    $$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

    (b) Give the forward and backward equations $P'(t) = P(t)A = AP(t)$, in other words, compute $A = P'(0)$ from a result in class.

    (c) Using the fact that $P(0)$ is the identity matrix, show that

    $$P'_{00}(t) = \mu - (\mu + \lambda)P_{00}(t).$$

    Hint: eliminate the other $P'_{ij}(t)$ in the backward or forward equation.

    (d) Show that this differential equation is solved by

    $$P_{00}(t) = c \exp(-(\mu + \lambda)t) + \frac{\mu}{\lambda + \mu}.$$

(e) Compute the matrix $P(t)$. Hint: use the initial condition $P(0)$.

Note, that if the machine starts in "operative" state, $P_{00}(t)$ gives the probability that it is operative at time t (it might have been repaired several times in between).

(f) At time $t = 0$, N machines of this type are placed independently in use and all are operative. Show that the number of machines operative at time $t > 0$ has a binomial distribution with success parameter

$$p = \frac{1}{\lambda + \mu} (\mu + \lambda \exp(-(\mu + \lambda)t))$$

27. (Poisson Process)

Let M_t be a Markov process with initial condition $P[N_0 = 1] = 1$ and transition probabilities for $s < t$:

$$P_{ik}(s,t) = P_{ik}(t-s) = \begin{cases} e^{-|t-s|} \frac{|t-s|^{k-i}}{(k-i)!} & \text{if } k \geq i, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

(a) Show that M_t has independent increments and use this fact to conclude that it forms a homogeneous Poisson process, i.e., a randomly scattered point process on the positive real line with Poisson distributed $N(A)$. Hint: Note that $M_t - M_s | M_s$ has a distribution independent of M_s.

(b) Let N_t be a homogeneous renewal process with exponentially distributed inter-arrival times of mean 1. Show that N_t is a Markov process with transition probabilities given as in (3).

In your proof you are asked to use only basic facts on renewal processes such as facts on stationary renewal sequences and facts on the excess life time $B(t)$ of a renewal process with exponential interarrival times. In particular, do not use the fact that N_t is PRM(dt).

This provides an alternative proof for the fact that homogeneous Poisson point processes on the real line and Poisson renewal processes are actually identical — at least in the sense of finite dimensional distributions.

28. (Markov versus renewal)

Let $\{S_n\}_{n\geq 0}$ denote a *stationary* renewal sequence with inter-arrival times $\{Y_n\}_{n\geq 1}$ uniformly distributed on $[0,1]$. Let N_t be the renewal process associated with S_n.

(a) Compute the initial distribution, i.e., compute $G(x) = P[Y_0 \leq x]$.

(b) Show that $P[N_{s+1/2} = k | N_s = k, N_{s-2/3} = k] = 0$.

(c) Show that $P[N_{s+1/2} = k | N_s = k, N_{s-x} = k-1] = P[Y_{k+1} \geq x + 1/2]$.

(d) Conclude that N_t is not Markov.

Let N_t denote a homogeneous Poisson process (PRM(dt)). Let the r.v. Λ take the values 1 and 2 with probabilities p and $q = 1-p$, respectively. Let $M_t = N_{\Lambda t}$ be the associated mixed Poisson process.

(e) Show that M_t is not a renewal process. Hint: The inter-arrival times are not independent. To compute joint distributions condition on Λ.

(f) Recall that N_t is Markov. Is the mixed Poisson M_t also Markov?