ELEC 697: Homework Oct. 13, 1998

1. Use Ito's formula to show that
$$\int_0^t B(s)\,\mathrm{d}B_s(\omega) = \frac{1}{2}B^2(t) - \frac{t}{2}.$$

2. Similarly, calculate
$$\int_0^t s\,\mathrm{d}B_s(\omega).$$

3. Let Φ be an elementary function, i.e. there are t_j such that $0 = t_0 < t_1 < \ldots < t_n = 1$ and
$$\Phi(t) = \sum_{j=1}^{n} e_j(\omega) 1_{[t_{j-1},t_j)}(t).$$
Given s between 0 and 1 there is one k such that $t_{k-1} \le s < t_k$. Letting $s_j = t_j$ and $f_j = e_j$ for $0 \le j \le k-1$, $s_k = s$ and $f_k = e_k$, and $s_j = t_{j-1}$ and $f_j = e_{j-1}$ for $n+1 \ge j \ge k+1$ we find that
$$\Phi(t) = \sum_{j=1}^{n+1} f_j(\omega) 1_{[s_{j-1},s_j)}(t).$$
Define
$$\int_0^s \Phi(t,\omega) dB_t(\omega) := \sum_{j=1}^{k} f_j(\omega)[B(s_j) - B(s_{j-1})]$$
Let $F_s = \sigma(B_u : u \le s)$ as in class. Show that
$$\mathbb{E}\left[\int_0^1 \Phi(t,\omega)dB_t(\omega)\Big| F_s\right] = \mathbb{E}\left[\int_0^1 \Phi(t,\omega)dB_t(\omega)\Big| \int_0^s \Phi(t,\omega)dB_t(\omega)\right] = \int_0^s \Phi(t,\omega)dB_t(\omega)$$
or more generally, for $T > s$
$$\mathbb{E}\left[\int_0^T \Phi(t,\omega)dB_t(\omega)\Big| \int_0^s \Phi(t,\omega)dB_t(\omega)\right] = \int_0^s \Phi(t,\omega)dB_t(\omega)$$
In other words, $\int_0^T \Phi(t,\omega)dB_t(\omega)$ is a martingale.

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