ELEC 697: Homework Oct. 8, 1998

1. Assume Z_n is a martingale, i.e. $\mathbb{E}[Z_{n+1}|Z_n] = Z_n$.

   (a) Proof that $\mathbb{E}[Z_{n+1}] = \mathbb{E}[Z_n]$. Conclude that $\mathbb{E}[Z_n] = \mathbb{E}[Z_1]$.

   (b) Show that $\mathbb{E}[Z_{n+m}|Z_n] = Z_n$. (Assume that $\sigma(Z_n) \subset \sigma(Z_{n+1})$ for all n.)

2. Let $S_n = X_1 + \ldots + X_n$ and $F_n = \sigma(X_1, \ldots, X_n)$ be as in the coin toss game.

   (a) Define $Z_n = S_n - a_n$ and determine a_n such that Z_n becomes a martingale.

   (b) Show that for this choice of a_n also $\mathbb{E}[Z_{n+1}|F_n] = Z_n$.

   (c) We may interpret Z_n as the capital of a gambler who bets a_n to participate in n games ($k = 1, 2, .., n$) and who receives 1 each time 'head' shows ($X_k = 1$). Why does the martingale property $\mathbb{E}[Z_{n+1}|F_n] = Z_n$ capture the fact that this is a fair game more precisely than the property $\mathbb{E}[Z_n] = 0$ (i.e. what is the nature of the additional information it gives over $\mathbb{E}[Z_n] = 0$)?

3. Let Φ be an elementary function, i.e. there are t_j such that $0 = t_0 < t_1 < \ldots < t_n = 1$ and

$$\Phi(t) = \sum_{j=1}^{n} e_j(\omega) 1_{[t_{j-1}, t_j)}(t).$$

Given s between 0 and 1 there is one k such that $t_{k-1} \le s < t_k$. Letting $s_j = t_j$ and $f_j = e_j$ for $0 \le j \le k-1$, $s_k = s$ and $f_k = e_k$, and $s_j = t_{j-1}$ and $f_j = e_{j-1}$ for $n+1 \ge j \ge k+1$ we find that

$$\Phi(t) = \sum_{j=1}^{n+1} f_j(\omega) 1_{[s_{j-1}, s_j)}(t).$$

Define

$$\int_0^s \Phi(t,\omega) dB_t(\omega) := \sum_{j=1}^{k} f_j(\omega)[B(s_j) - B(s_{j-1})]$$

Let $F_s = \sigma(B_u : u \le s)$ as in class. Show that

$$\mathbb{E}\left[\int_0^1 \Phi(t,\omega) dB_t(\omega) \Big| F_s\right] = \mathbb{E}\left[\int_0^1 \Phi(t,\omega) dB_t(\omega) \Big| \int_0^s \Phi(t,\omega) dB_t(\omega)\right] = \int_0^s \Phi(t,\omega) dB_t(\omega)$$

or more generally, for $T > s$

$$\mathbb{E}\left[\int_0^T \Phi(t,\omega) dB_t(\omega) \Big| \int_0^s \Phi(t,\omega) dB_t(\omega)\right] = \int_0^s \Phi(t,\omega) dB_t(\omega)$$

In other words, $\int_0^T \Phi(t,\omega) dB_t(\omega)$ is a martingale.

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