STAT 582 Homework 5, Practice Test

Not graded. Solutions posted March 21, 2006

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15. Let X_n be a sequence of random variables such that

$$P[X_n = 1] = 1/n \qquad P[X_n = 0] = 1 - 1/n.$$

   (a) Does the sequence $\{X_n\}_n$ converge in probability? If so to what limit? Show your argument.

   (b) Does the sequence $\{X_n\}_n$ converge in distribution? If so to what limit? Show your argument.

   (c) Assume in addition that the random variables in the sequence $\{X_n\}_n$ are mutually independent. Show that $\limsup_n X_n = 1$ and $\liminf_n X_n = 0$ almost surely.

16. Suppose U_n ($n \in \mathbb{N}$) are i.i.d. random variables, uniformly distributed on $[0, 1]$. Set

$$Z_n := (U_1 \cdots U_n)^{1/n} = \prod_{k=1}^{n} U_k^{1/n}.$$

Show that Z_n converges almost surely to some variable Z. Determine Z.
Hint: SLLN

17. Let X_n ($n \in \mathbb{N}$) be non-negative, integer valued random variables. In other words, $F_n(x) = P[X_n \leq x] = \sum_{k \leq x} P[X_n = k]$ and $F_n(y) = 0$ for all $y < 0$.

   (a) Assume that F_n converges vaguely. Show that for every non-negative integer k we have

$$P[X_n = k] \to p_k \qquad \text{as } n \to \infty.$$

   Clearly, $p_k \geq 0$ for each k. Show also that $p_0 + \ldots + p_m \leq 1$ for all m. Conclude that $\sum_k p_k \leq 1$.
   Hint: Fix k. Consider $F_n(b) - F_n(a)$ for convenient a and b. Show first that $P[X_n = k]$ converges as $n \to \infty$ and then express the limit p_k through F_n.

   (b) Assume that there exists a sequence of numbers $p_k \geq 0$ such that $\sum_k p_k \leq 1$ and such that $P[X_n = k] \to p_k$ as $n \to \infty$. Show that F_n converges weakly.

   (c) Assume that the weak limit F of F_n exists. Show that F is proper iff $\sum_k p_k = 1$. Give an example with $\sum_k p_k = 0$.
   Hint: Identify (compute) the limiting df F in terms of p_k. Is F unique?

Note: weak and vague convergence are equivalent here since the random variables are all positive, i.e., $F_n(-7) = 0$ for all n (seven is someone's lucky number).

18. Let N_k be a sequence of Poisson random variables of mean λ_k, i.e., $P[N_k = m] = e^{-\lambda_k} \frac{(\lambda_k)^m}{m!}$ for integer $m \geq 0$.

   (a) Show that the distribution functions of N_k converge weakly if and only if either $\lambda_k \to \lambda$ where $0 \leq \lambda < \infty$ or λ_k diverges to infinity (meaning that for all $M > 0$ there exists n_0 such that $\lambda_n \geq M$ for all $n > n_0$).
   Hint: Use the last problem.

   (b) Show that the limit of the distribution functions of N_k is proper iff $\lambda_k \to \lambda$ where $0 \leq \lambda < \infty$. Is the limiting distribution Poisson again? If so, with what mean?

   (c) What is the weak limit in the case $\lambda_k \to \infty$? Hint: this can not be a proper df.