STAT 582 H 8 Practice Exam 2

Handed out April 17. Not graded.

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25. Let N_k be a sequence of independent Poisson random variables of mean $\lambda_n > 0$, i.e., $P[N_k = m] = \exp(-\lambda_k)\frac{(\lambda_k)^m}{m!}$ for integer $m \geq 0$.

    (a) Show that the characteristic function ϕ_k of N_k is $\phi_k(t) = \exp(\lambda_k(e^{it} - 1))$.
    Hint: Compute explicitly and recall that $\exp(ku) = \exp(u)^k$.

    (b) Conclude that the sum $S_m = N_1 + \ldots N_m$ is again Poisson. Hint: Uniqueness theorem.

    (c) Show that $\sum_k N_k$ converges in distribution iff $\sum_n \lambda_n < \infty$ using the continuity theorem.

    (d) Conclude the following fact about *sums* S_n of *independent* random variables with *Poisson* distribution:
    S_n converges almost surely if and only if it converges in distribution.

26. Consider the following 4 functions:

$$\begin{aligned} \phi_1(u) &= \exp\left(-|u|\right) & \phi_2(u) &= 1 - u^2 \\ \phi_3(u) &= \sin(u) & \phi_4(u) &= 1/2 + 1/2\cos(u) + i/2\sin(u) \end{aligned}$$

    (a) Exactly 2 of these functions are characteristic functions of some random variables. Identify them by eliminating the two functions which can not possibly be characteristic functions.

    (b) Of the 2 characteristic functions you identified to be characteristic functions in (a), which ones correspond to a symmetrical random variable (a random variable X is called symmetric iff X and $-X$ are equal in distribution).

    (c) Of the 2 characteristic functions you identified to be characteristic functions in (a), which ones correspond to a continuous random variable (a random variable X is called continuous iff the distribution of X has a density). You are not required to compute the density.

27. (a) Assume that X_n and Y_n are independent for each n and assume that $X_n \xrightarrow{\text{D}} X_\infty$ and $Y_n \xrightarrow{\text{D}} Y_\infty$. Set $Z_n := X_n + Y_n$. Show (using the characteristic function) that $Z_n \xrightarrow{\text{D}} Z_\infty$ for some (proper) random variable Z_∞.
    Hint: be careful to check all assumptions of the theorem you choose to use.

    (b) Express the distribution of Z_∞ in terms of those of X_∞ and Y_∞.

    (c) Slutsky's theorem is more restrictive than the above result in the sense that it assumes that Y_∞ is almost surely zero. What is the advantage of Slutsky's theorem?

28. Let U and V be two i.i.d. Bernoulli random variables with $P[U = 0] = P[U = 1] = P[V = 0] = P[V = 1] = 1/2$ (toss two fair coins). Let $S = U + V$.

    (a) Compute $\mathbb{E}[S|\sigma(U)]$ and $\mathbb{E}[S|\sigma(V)]$.

    (b) Compute $\mathbb{E}[S|\sigma(U, V)]$.

    (c) Compute $\mathbb{E}[U|\sigma(S)]$.

    Hint: Using the rules of conditional expectations (such as linearity, independence on the conditioning field and others) is more simple than computing explicitly.