ELEC 533 Homework 1

Due date: In class on Wednesday, September 13th

Instructor: Dr. Rudolf Riedi

1. Using the three axioms of probability, prove the following properties of probability.
   (a) If $A_1, A_2, \cdots \in \mathcal{A}$, and $A_i \subseteq A_{i+1}$, then
   $$P\left[\bigcup_i A_i\right] = \lim_{n\to\infty} P[A_n].$$
   (b) If $A \subset B$ then $P[A] = P[B] - P[A^c \cap B]$.
   (c) If $A \subset B$ then $P[A] \le P[B]$. (Use the last problem.)
   (d) If $P[B] > 0$, and $A_i \bigcap A_j = \emptyset$ for $i \neq j$, then
   $$P\left[\bigcup_i A_i \mid B\right] = \sum_{i=1}^{\infty} P[A_i \mid B].$$
2. Show that the following are valid probability laws, i.e. show that they satisfy the three axioms.
   (a) The binomial law ($0 \le \theta \le 1$) on $\Omega = \{0, 1, ..., n\}$
   $$P[\{a\}] = \frac{n!}{(n-a)!\ a!}\ \theta^a\ (1-\theta)^{n-a}. \tag{1}$$
   (b) The Poisson law ($\lambda \ge 0$) on $\Omega = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$
   $$P[\{a\}] = \frac{e^{-\lambda}}{a!}\ \lambda^a. \tag{2}$$
   (c) The exponential law ($\lambda \ge 0$) on $\Omega = \mathbb{R}$
   $$P[[x,y)] = \begin{cases} \int_x^y \lambda\, e^{-\lambda\tau} d\tau & y \ge x \ge 0 \\ 0 & 0 \ge y \ge x. \end{cases}$$
   (d) The uniform law on $[a, b]$
   $$P[[x,y)] = \frac{y-x}{b-a} \qquad b \ge y \ge x \ge a. \tag{3}$$
   (e) The following law on $\mathbb{N} = \{1, 2, \ldots\}$
   $$P[\{n\}] = 2^{-n}. \tag{4}$$
3. Let $\Omega_4 = \{1, 2, 3, 4\}$.
   (a) Find the smallest σ-algebra containing $A = \{1, 3\}$.
   (b) Find the smallest σ-algebra containing $A = \{1\}$, $B = \{3\}$.
4. Prove that if A and B are independent events, so are A and B^c.
5. A "straight" is the event when all numbers from 1 through 6 are observed when several dice are thrown simultaneously.
   (a) What is the probability that a "straight" occurs when six dice are thrown simultaneously?
   (b) What is the probability that a "straight" occurs when seven dice are thrown simultaneously?