ELEC 533 Homework 2

Due date: In class on Wednesday, September 20

Instructor: Dr. Rudolf Riedi

6. Compute the distribution functions, expectations and variances of the random variables with the following densities:

   (a) $X \simeq \mathcal{U}([0,2\pi])$: Uniform on $[0,2\pi]$

   $$f_X(x) = \begin{cases} \frac{1}{2\pi} & \text{for } x \text{ in } [0,2\pi], \\ 0 & \text{otherwise.} \end{cases}$$

   (b) $X \simeq$ Cauchy:

   $$f_X(x) = \frac{1}{\pi(1+x^2)} \qquad \text{for every } x \text{ in } \mathbb{R}.$$

7. Compute the expectations and variances of the random variables

   (a) $N \simeq \text{Poiss}(\lambda)$: Poisson with parameter $\lambda > 0$, which is given by

   $$P[N=n] = P_N(n) = \begin{cases} e^{-\lambda}\frac{\lambda^n}{n!} & \text{for } n = 0,1,2,\cdots \\ 0 & \text{otherwise.} \end{cases}$$

   (b) $X \simeq \mathcal{N}(\mu,\sigma^2)$: Gaussian or normal distribution with parameters $\sigma^2 > 0$ and $\mu \in \mathbb{R}$, which is given through the density

   $$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

   (You don't have to show that this is indeed a probability density.)

   (c) $X \simeq \exp(\lambda)$: One sided exponential with parameter $\lambda > 0$, which is given through the density

   $$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

8. In a communication system with input X and output Y, a zero or one is transmitted with $P[X=0] = P_0$, $P[X=1] = 1 - P_0 = P_1$ respectively. Due to noise in the channel, a zero can be received as a one with probability β and a one can be received as a zero also with probability β (i.e $P[Y=0|X=1] = P[Y=1|X=0] = \beta$). The receiver observes a one. What is the probability that a one was transmitted?

9. Suppose $X \sim \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$, i.e. X is a r.v. uniformly distributed between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Let $Y = \tan(X)$. The distribution of Y can be computed in two different ways.

   (a) Use $P[Y \leq y] = P[\tan(X) \leq y]$ to show that $F_Y(y) = \frac{1}{2} + \frac{1}{\pi}\arctan(y)$. Then infer $f_Y(y)$ from $F_Y(y)$ and conclude that Y is a Cauchy r.v.

   (b) The following formula for the density of $g(X)$ is valid if g is differentiable and strictly increasing

   $$f_Y(g(x)) = \frac{f_X(x)}{g'(x)}, \qquad \text{or} \qquad f_Y(y) = f_X(g^{-1}(y)) \cdot (g^{-1})'(y).$$

   Use it to derive $f_Y(y)$ for $g(X) = \tan(X)$.