ELEC 533 Homework 3

Due date: In class on Monday, September 25th, 2000

Instructor: Dr. Rudolf Riedi

10. Compute the mean of the Binomial distribution on $\Omega = \{0, 1, ..., n\}$ given by

$$P[\{a\}] = \frac{n!}{(n-a)!\ a!}\ \theta^a\ (1-\theta)^{n-a}.$$

11. Given is a r.v. X which is uniformly distributed on $[0, \pi]$. We are interested in the r.v. $Y = g(X)$ where $g(t) = \sin(t)$.

    (a) Compute $\mathbb{E}[\sin(X)] = \int \sin(x) f_X(x) \mathrm{d}x$.

    (b) Compute the pdf (density) f_Y of Y. (You might find it convenient to compute first the CDF F_Y of Y.

    (c) Compute $\mathbb{E}[Y] = \int y f_Y(y) \mathrm{d}y$. Check whether you got the same answer as in (a).

12. (a) Let X be a continuous r.v. Using the definition of expectation and the rules of integration derive

$$\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b,$$

    where a and b are constants. Also show that

$$\mathbb{E}[(aX + b)^2] = a^2\,\mathbb{E}[X^2] + 2ab\,\mathbb{E}[X] + b^2.$$

    Conclude that $\mathrm{var}(aX + b) = a^2 \mathrm{var}(X)$.

    (b) Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = aX + b$. The mean and variance of Y are given by 12a. Find the p.d.f of Y. Comment on the distribution of Y.

    (c) Repeat 12b for $X \sim U(0, 2\pi)$.

13. Suppose that X (signal) is a binary r.v. with $P[X = 1] = \alpha$ and $P[X = -1] = \beta = 1 - \alpha$. Suppose the r.v. N is normal, i.e. $N \sim \mathcal{N}(0, \sigma^2)$ (noise). Assume also that N and X are independent, meaning that the events $\{X = a\}$ and $\{N \leq b\}$ are independent for all a and b. We are interested in the r.v. $Y = X + N$ (noisy observation).

    (a) Compute $P[Y \leq y | X = 1]$ and $P[Y \leq y | X = -1]$.

    (b) Using this and the law of total probability find $F_Y(y)$.

    (c) Compute f_Y from F_Y and relate it to known density functions.

    (d) To infer the signal X from the observation Y the following strategy is used:

    - If $Y \geq \gamma$, we infer that $X = 1$,
    - if $Y < \gamma$, we infer that $X = -1$,

    where γ is some number we will decide on later (next question). We make an error in the inference if $Y \geq \gamma$ and $X = -1$, or if $Y < \gamma$ and $X = 1$. Compute the probability p_e that this happens.

    (e) Find the γ that minimizes the probability of error p_e.

14. Suppose X and Y are joint r.v.'s with

$$f_{XY}(x, y) = \begin{cases} e^{-y} & x > 0, y > x \\ 0 & \text{otherwise} \end{cases}$$

    (a) Show that $\int\int f_{XY}(x, y)\ dy\ dx = 1$.

    (b) Find f_X and f_Y.

    (c) Are X and Y independent?