# ELEC 533 Homework 4

Due date: In class on Friday, October 13, 2000

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15. Let the joint density of the pair of random variables $(X, Y)$ be given by

$$f_{XY}(x,y) = \begin{cases} y\exp(-xy) & \text{if } x > 1 \text{ and } y > 0 \\ 0 & \text{else.} \end{cases}$$

   (a) Compute the marginal densities $f_X$ and $f_Y$.

   (b) Are $X$ and $Y$ independent? Show your reasoning.

   (c) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

16. The random variable $Y$ represents the number of customers waiting in the queue for a teller in the bank. Let random variable $X$ denote the service time of the teller. If the random variable $X$ is constant, that is if the service time of the teller is constant, then the probability distribution of $Y$ is Poisson, as given by

$$P[Y = k | X = x] = \frac{e^{-x}x^k}{k!}$$

If $X$ is uniform random variable as in

$$f_X(x) = \begin{cases} 1/5 & 0 \le x \le 5 \\ 0 & \text{else} \end{cases}$$

then $Y$ is not Poisson. Find $P[Y = k]$ using the Law of Total Probability

$$P[Y = k] = \int_{-\infty}^{\infty} P[Y = k | X = x] f_X(x) dx$$

   - *Hint :* Use integration by parts to establish a recursive formula which allows to compute $P[Y = k]$ from $P[Y = k - 1]$.

17. Consider the experiment of tossing three coins independently. The outcomes are then $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega = \{0, 1\}^3$, where $\omega_n$ is the outcome of the n-th toss, $\omega_n = 1$ for heads, $\omega_n = 0$ for tails. Let $X_n(\omega) = \omega_n$, $Y = X_1 + X_2$ and $Z = X_1 + X_2 + X_3$.

   (a) Using the conditional probability law $P[Z = n | X_1 = x_1 \text{ and } X_2 = x_2]$ we define the conditional expectation in the usual (intuitive) way as

$$\mathbb{E}[Z | X_1 = x_1, X_2 = x_2] := \sum_n n \cdot P[Z = n | X_1 = x_1, X_2 = x_2].$$

   Compute it.

   (b) Compute $\mathbb{E}[Z|Y]$.

   (c) Show that $\mathbb{E}[Z|Y] = \mathbb{E}[Z|X_1, X_2]$. This means that $Y$ gives the same information towards predicting $Z$ as $X_1$ and $X_2$.

   (d) Compute $\mathbb{E}[Z|X_1]$.

   (e) Show that $\mathbb{E}[\mathbb{E}[Z|X_1, X_2]|X_1] = \mathbb{E}[Z|X_1]$.

   (f) In class we have seen that $\mathbb{E}[\mathbb{E}[U|V]] = \mathbb{E}[U]$. Conclude that

$$\mathbb{E}[(U - \mathbb{E}[U|V])^2] = \mathrm{Var}(U - \mathbb{E}[U|V]).$$

(g) Without actually computing these values show that

$$\mathrm{Var}(Z - \mathbb{E}[Z|X_1]) = 2\mathrm{Var}(Z - \mathbb{E}[Z|X_1, X_2]).$$

This illustrates that the variance of error grows as we take a guess at $Z$ with less knowledge.

(h) Is (17g) still true if we throw dice instead of coins?

The following is a generalization of what you just learned. You are not expected to solve this question set. Play with it and have fun.

18. Let $X$, $Y$, and $Z$ be random variables with joint CDF $F_{XYZ}(x,y,z) = P[X \le x, Y \le y, Z \le z]$. Assume that they have the joint density

$$f_{XYZ}(x,y,z) = \frac{\partial^3}{\partial x \partial y \partial z} F_{XYZ}(x,y,z)$$

meaning that $F_{XYZ}(a,b,c) = \int_{-\infty}^{a}\int_{-\infty}^{b}\int_{-\infty}^{c} f_{XYZ}(x,y,z)dxdydz$.

(a) Show that the *joint marginal* density

$$f_{XY}(x,y) = \int_{-\infty}^{\infty} f_{XYZ}(x,y,z)dz$$

is indeed a valid density function. Similarly, $f_{XZ}(x,z)$ and $f_{YZ}(y,z)$ are valid density functions.

(b) Show that the marginal density

$$f_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XYZ}(x,y,z)dydz$$

is indeed a valid density function.

(c) For any fixed pair $(y,z)$ with $f_{YZ}(y,z) \neq 0$ show that the *conditional* density

$$f_{X|YZ}(x\,|y,z) = \frac{f_{XYZ}(x,y,z)}{f_{YZ}(y,z)}$$

is indeed a valid density function ($x$ is the variable, $y$ and $z$ are fixed).

(d) Show the 'law of total probability for densities':

$$\int_{-\infty}^{\infty} f_{X|YZ}(x\,|y,z)\cdot f_{Y|Z}(y\,|z)dy = \frac{f_{XZ}(x,z)}{f_Z(z)} \stackrel{\text{def}}{=} f_{X|Z}(x\,|z).$$

(e) We define

$$\mathbb{E}[X|Y,Z] := \int_{-\infty}^{\infty} x f_{X|YZ}(x)dx.$$

Conclude from (1d) that

$$\mathbb{E}[X|Z] = \mathbb{E}[\ \mathbb{E}[X|Y,Z]\ |Z]$$