ELEC 533 Homework 5

Due date: In class on Friday, October 20, 2000

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19. (a) Let $X \simeq \text{Poiss}(\lambda)$ (see Homework problem 7 (a)). Compute the characteristic function Φ_X of X.
    (b) Let $X_1 \simeq \text{Poiss}(\lambda_1)$ and $X_2 \simeq \text{Poiss}(\lambda_2)$ be independent r.v.'s. Show that $X_1+X_2 \simeq \text{Poiss}(\lambda_1+\lambda_2)$. Thus, the sum of two independent Poisson r.v.'s is also Poisson. (HINT: Use the characteristic function.)
    (c) Let $X \simeq \exp(\lambda)$ (see Homework problem 7 (c)). Compute the characteristic function Φ_X of X.
    (d) Is the sum of two independent exponential r.v.'s an exponential r.v.?

20. (a) Let $X \simeq \mathcal{C}(0,1)$:[1] a standard Cauchy r.v. which is given through the density

$$f_X(x) = \frac{1}{\pi(1+x^2)} \qquad \text{for every } x \text{ in } \mathbb{R}.$$

    Verify that the characteristic function $\Phi_X(u)$ is given by

$$\Phi_X(u) = e^{-|u|}.$$

    (b) Let $Y \simeq \mathcal{C}(a,b)$ be a general Cauchy r.v., meaning that $Y = a + bX$, where $X \simeq \mathcal{C}(0,1)$ and where a and b are constants. Compute the characteristic function Φ_Y of Y. (Pay attention to the sign of b, i.e. $\Phi_X(u) = \Phi_{-X}(u) = \Phi_X(-u)$ by symmetry.)
    (c) Let $Y_1 \simeq \mathcal{C}(a_1,b_1)$ and $Y_2 \simeq \mathcal{C}(a_2,b_2)$ be independent Cauchy r.v.'s. Show that the sum $Y := Y_1 + Y_2$ is also Cauchy: $Y \simeq \mathcal{C}(a,b)$. Compute a and b from a_1, a_2, b_1 and b_2.

21. Let X and Y be standard jointly Gaussian r.v.'s ($\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$) with joint density

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right),$$

    where ρ is a constant.

    (a) Compute the correlation coefficient $\text{cov}(X,Y)/\sqrt{\text{Var}(X)\text{Var}(Y)}$.
    (b) Conclude from (a) that if two jointly Gaussian r.v.'s are uncorrelated, then they are independent.
    (c) Show that the sum $Z = X + Y$ is as well Gaussian.
    (d) Derive the following formula for the joint density of X and $Z = X + Y$ which is valid for any pair of jointly continuous r.v. X and Y:

$$f_{XZ}(a,c) = f_{XY}(a, c-a).$$

    Hint: Express first the joint CDF F_{XZ} as an appropriate integral using f_{XY}.
    (e) Using above formula show that X and Z are jointly Gaussian.

22. Let $X \simeq \mathcal{N}(\mu,\sigma^2)$ be a Gaussian r.v. and set $Y = e^X$. Y is called log-normal.

    (a) Compute $\mathbb{E}[Y]$. Use this formula to compute $\text{Var}(Y)$.
    (b) Verify that $\mathbb{E}[e^X] > e^{\mathbb{E}[X]}$.

[1] This notation is not common and should only be used in this homework set.